On the non-existence of even degree graphs with diameter 2 and defect 2

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Abstract

Using eigenvalue analysis, it was shown by Erdös et al. that, with the exception of C_4, there are no graphs of diameter 2, maximum degree d and d^2 vertices. In this paper, we show that graphs of diameter 2, maximum degree d and d^2-1 vertices do not exist for most values of d, when d is even, and we conjecture that they do not exist for any even d greater than 4.

Keywords: Moore graphs; diameter 2; degree/diameter problem

1 Introduction

There are many famous and difficult graph-theoretical problems that arose over the past four decades from the design of interconnection networks (such as local area networks, parallel computers, switching system architecture in VLSI technology, and many others). Perhaps one of the most prominent problems is the *degree/diameter problem* which is to determine, for each d and k, the largest order $n_{d,k}$ of a graph of maximum degree d and diameter at most k. It is easy to show that $n_{d,k} \leq M_{d,k}$ where $M_{d,k}$ is the *Moore bound*, given by

$$n_{d,k} \leq M_{d,k} = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$$

For a survey of the degree/diameter problem, see Miller et al. (2005).

In this paper we concentrate on the case when the diameter is equal to 2. Since a graph of diameter 2 and maximum degree d may have at most d^2+1 vertices, it was asked in (Erdös et al. 1980): Given non-negative integer numbers d and Δ (*defect*), is there a graph of diameter 2 and maximum degree d with $d^2+1-\Delta$ vertices? It was proved in (Hoffman et al. 1960) that if $\Delta = 0$ then there are unique graphs corresponding to $d = 2, 3, 7$ and possibly $d = 57$. The case $\Delta = 1$ was solved by Erdös et al. (Erdös et al. 1980). In this paper we consider the case $\Delta = 2$ and prove that graphs of defect 2 do not exist for most values of degree d in the case when d is even.

We refer to a graph of maximum degree d, diameter $k \geq 2$ and order $M_{d,k} - \Delta$ ($\Delta \geq 1$) as a (d, k, Δ)-*graph*. Let G be a (d, k, Δ)-graph.

Definition 1 Let u be a vertex in G. A vertex v in G is called a *repeat* of u with *multiplicity* $m_v(u)$ ($1 \leq m_v(u) \leq \Delta$) if there are exactly $m_v(u)+1$ different paths of lengths at most k from u to v.

It is immediate that

Observation 1 *Vertex u is a repeat of v with multiplicity $m_u(v)$ if and only if v is a repeat of u with the same multiplicity.*

A repeat with multiplicity 1 will be called a *single* repeat, a repeat with multiplicity 2 will be called a *double* repeat, a repeat with multiplicity Δ will be called a *maximal* repeat.

We denote by $R_s(u)$ the set of all repeats of a vertex u in G. Taking into account the multiplicities of repeats, we denote by $R_m(u)$ the multiset of all the repeats of a vertex u in G, containing each repeat v of u exactly $m_v(u)$ times.

Let u be a vertex in G. We denote by $N(u)$ the set of all neighbours of u. If A is a multiset of vertices of G, then $N(A)$ denotes the multiset of all the neighbours of the vertices of A. We use $R_m(A)$ to denote the multiset of all the repeats of all vertices in A.

Proposition 1 *If G is regular then, for all $u \in V(G)$,*

$$|R_m(u)| = \sum_{v \in R_s(u)} m_v(u) = \Delta.$$

□

Definition 2 A subset S of $V(G)$ is called a *closed repeat set* if $R_m(S) = S$. A closed repeat set is *minimal* if none of its proper subsets is a closed repeat set.

Definition 3 A *repeat subgraph* H_S of a closed repeat set S of G is a multigraph whose vertex set $V(H_S) = S$ and the number of parallel edges between a vertex u and any of its repeats, say $v \in R_m(u)$, equals the multiplicity $m_v(u)$.

We observe that

Observation 2 *If* $\Delta < 1 + (d-1) + \ldots + (d-1)^{k-1}$ *then* G *is regular.*

It is also true that

Observation 3 *If* G *is regular then the repeat graph* H_G *of* G *is* Δ*-regular.*

Note that instead of writing "a vertex x is adjacent to a vertex y" we write $x \sim y$, and if x is not adjacent to y then we write $x \nsim y$. Unless explicitly shown where necessary, by u_i and u_j $(i \neq j)$ we shall mean two distinct vertices.

2 Structural properties of $(d, 2, 2)$-graphs

In this section we consider graphs of diameter 2 with defect 2. Such graphs do not exist for $d \leq 2$. Let G be a $(d, 2, 2)$-graph for $d \geq 3$. From Observation 2, we have that

Observation 4 *Every* $(d, 2, 2)$*-graph for* $d \geq 3$ *is regular.*

Let us consider repeat configurations in $(d, 2, 2)$-graphs. Let u be a vertex of a $(d, 2, 2)$-graph. Then there are two possibilities:

- u has two single repeats, $r_i(u)$, $i = 1, 2$.
- u has one double (maximal) repeat, $r(u) = r_1(u) = r_2(u)$, with multiplicity 2.

With respect to repeats in G, there are five possible repeat configurations, as depicted in Fig. 1.

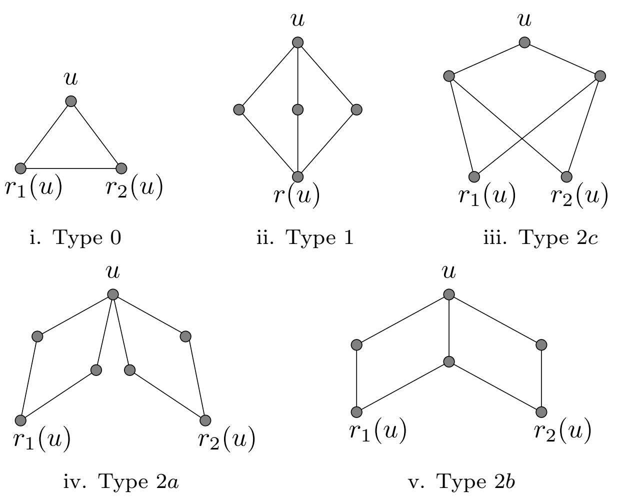

Figure 1: Possible repeat configurations for vertex u in a $(d, 2, 2)$-graph.

We will denote the set of vertices of each type by Type 0, Type 1, Type 2a, Type 2b and Type 2c, as shown in Fig. 1. We denote by n_0, n_1, n_{2a}, n_{2b}, n_{2c} the number of vertices of the corresponding repeat types.

Fig. 2 shows the only known $(d, 2, 2)$-graph (for even d) whose uniqueness is shown in (Broersma et al. 1988).

We observe the following

Observation 5 $n_0 + n_1 + n_{2a} + n_{2b} + n_{2c} = d^2 - 1$.

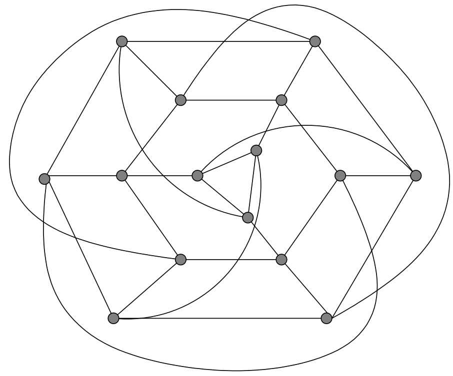

Figure 2: The only known $(d, 2, 2)$-graph for even d.

For the purpose of this paper, we shall consider each pair of parallel edges in H_G as a cycle of length 2.

Observation 6 H_G *is the union of cycles of lengths* ≥ 2*, each cycle a minimal closed repeat set of* G*.*

From now on, each cycle in H_G will be called a *repeat cycle*.

The following structural properties of G were proved in (Nguyen et al. 2007).

Theorem 1 *(Nguyen et al. 2007) In a* $(d, 2, 2)$*-graph* G*, if* d *is even then* $n_0 = 3$ *and* $n_{2b} = d^2 - 4$*.*

Corollary 1 *(Nguyen et al. 2007)* $n_{2b} \equiv 0 \pmod 2$*.*

Let the vertices u_0, u_1, u_2 form a triangle in G, denoted by T, and let Υ_{2b} be the subset of all vertices of type $2b$ in $N(u_0) \bigcup N(u_1) \bigcup N(u_2)$. Then Υ_{2b} is a minimal closed repeat set. We shall call Υ_{2b} the *outer repeat cycle* of T in H_G. Note that Υ_{2b} is the set of vertices at distance 1 from T, and $\Upsilon_{2b} \bigcap T = \emptyset$. The number of vertices of Υ_{2b} is $3(d-2)$.

Fig. 3 illustrates a labeled partial structure of G, in the case when d is even, which shows the cycle $u_0u_1u_2$ and its outer repeat cycle. Since Υ_{2b} contains all vertices of type $2b$ and Υ_{2b} is a minimal closed repeat set, by Corollary 1, there exists in H_G another cycle Υ'_{2b}, also of the same size as Υ_{2b}, that is, $3(d-2)$. Note that, in Fig. 3, $u_3 \sim u_{3d-4}$ and $u_{3d-3} \sim u_{9d-16}$. This is because u_3 and u_{9d-16} belong to Υ_{2b} whereas u_{3d-3} and u_{3d-4} belong to Υ'_{2b}.

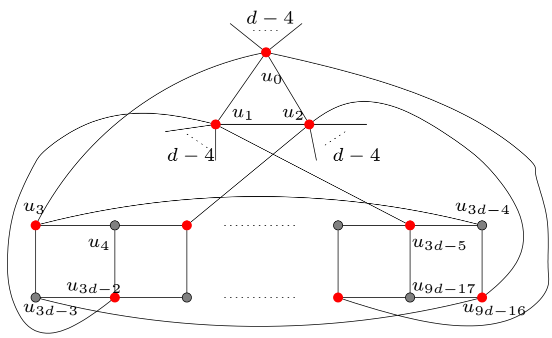

Figure 3: An illustration of the neighbourhood of T in G for even d.

Lemma 1 *(Nguyen et al. 2007) Let* T *be a triangle in* G *and let* Υ_{2b} *be the outer repeat cycle of* T *in* H_G*. Let* C_t *be any repeat cycle in* H_G *of length* $t \geq 4$ *such that there exists in* G *an edge between a vertex on* Υ_{2b} *and a vertex on* C_t*. Then either* $t = \frac{1}{3}|\Upsilon_{2b}|$ *or* $t \equiv 0 \pmod{|\Upsilon_{2b}|}$*.*

3 On the non-existence of $(d, 2, 2)$-graphs for even d

In this section, we shall prove that for most values of even d, $(d, 2, 2)$-graphs do not exist.

From Theorem 1, it immediately follows that

Corollary 2 *G is not vertex-transitive for even degree d.*

Lemma 2 *For even $d \geq 6$, every cycle, other than the triangle in H_G, has length $3k(d-2)$, for some $k \geq 1$.*

Proof. As demonstrated in Section 2, H_G contains one cycle of length 3 and at least two cycles of length $3(d-2)$. Let $u_1u_2u_3$ be the triangle T of G and let $v_1 \dots v_{3(d-2)}$ be the outer repeat cycle Υ_{2b} of T in H_G such that the repeats of v_j $(1 \leq j \leq 3(d-2))$ are $v_{(j-1)(\bmod (3(d-2)))}$ and $v_{(j+1)(\bmod (3(d-2)))}$. Without loss of generality, let us suppose that $u_1 \sim v_1$ and $u_2 \sim v_2$ in G.

Let the a_i, $i = 1, \dots, b$, be the lengths of the cycles in H_G and let $a_1 = 3$, $a_2 = a_2 = 3(d-2)$ correspond to T, Υ_{2b} and Υ'_{2b}, respectively. Thus, $f = \sum_{i=4}^{b} a_i = (d-2)(d-4)$.

Let C_{a_j} be an arbitrary cycle in H_G $(j \neq 1, 2, 3)$. Then, by Lemma 1, either $a_j = d-2$, or $a_j \equiv 0 \pmod{3(d-2)}$. Suppose that $a_j = d-2$. Denote by $w_1, \dots, w_{d-2}$ the vertices of C_{a_j} such that the repeats of w_k $(1 \leq k \leq d-2)$ are $w_{(k-1)\ (\bmod (d-2))}$ and $w_{(k+1)\ (\bmod (d-2))}$.

We know that the vertices of C_{a_j} must reach the vertices of T through the vertices of Υ_{2b}. Without loss of generality, suppose that $w_1 \sim v_1$ and $w_2 \sim v_2$. However, since $(d-2)$ is not divisible by 3 when d is even, by the Neighbourhood Theorem, u_1 and w_1 would then have at least three common neighbours, namely v_1, v_{d-1} and v_{2d-3}. This is clearly impossible.

Therefore, each a_i $(4 \leq i \leq b)$ must be a multiple of $3(d-2)$. □

Theorem 2 *For even $d \geq 4$, if $d \not\equiv 1 \pmod 3$ then there is no $(d, 2, 2)$-graph.*

Proof. Let b be the number of cycles in H_G. Let a_i, for $i = 1 \dots b$, be the lengths of these cycles, denoting by a_1 the triangle. Then as $\sum_{i=1}^{b} a_i = d^2 - 1$, by Lemma 2, we have that $\sum_{i=2}^{b} a_i = 3(d-2)k = d^2 - 4 = (d-2)(d+2)$. Therefore, $d + 2 \equiv 0 \pmod 3$. □

By counting the total number N_5 of 5-cycles in G, we derive some further necessary conditions for the existence of G.

Theorem 3 *For even $d \geq 4$, if $N_5 = \frac{(d-2)(d^4+2d^2-2d-25)}{10}$ is not an integer then there is no $(d, 2, 2)$-graph.*

The results of Theorems 2 and 3 improve the upper bound for the order of $(d, 2, 2)$-graphs so that $n_{d,2} \leq d^2 - 3$ for infinitely many even degrees d. For $d \geq 10$, the first 50 values of d for which G might still exist are shown in Table 1.

We conclude this paper by posing the following

Conjecture 1 *For even $d \geq 6$, $(d, 2, 2)$-graphs do not exist.*

10	22	34	40	52	64	70	82	94	100
112	124	130	142	154	160	172	184	190	202
214	220	232	244	250	262	274	280	292	304
310	322	334	340	352	364	370	382	394	400
412	424	430	442	454	460	472	484	490	502

Table 1: The first 50 values of d for which a $(d, 2, 2)$-graph might still exist for even d.

4 Acknowledgement

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