An open problem: $(4; g)$-cages with odd $g \geq 5$ are tightly superconnected

Jianmin Tang[1]*, Camino Balbuena[2], Yuqing Lin[3], Mirka Miller[1,4]

[1]**School of Information Technology and Mathematical Sciences**
University of Ballarat, Ballarat, Victoria 3353, Australia
jmtang@students.ballarat.edu.au, m.miller@ballarat.edu.au

[2] **Departament de Matemàtica Aplicada III**
Universitat Politècnica de Catalunya, Barcelona, Spain
m.camino.balbuena@upc.edu

[3] **School of Electrical Engineering and Computer Science**
The University of Newcastle, NSW 2308, Australia
yuqing.lin@newcastle.edu.au

[4] **Department of Mathematics**
University of West Bohemia, Plžen, Czech Republic

Abstract

Interconnection networks form an important area which has received much attention, both in theoretical research and in practice. From theoretical point of view, an interconnection network can be modelled by a graph, where the vertices of the graph represent the nodes of the network and the edges of the graph represent connections between the nodes in the network. Fault tolerance is an important performance feature when designing a network, and the connectivity of the underlying graph is one of the measures of fault tolerance for a network.

A graph is connected if there is a path between any two vertices of G. We say that G is *t-connected* if the deletion of at least t vertices of G is required to disconnect the graph. A graph with minimum degree δ is *maximally connected* if it is δ-connected. A graph is *superconnected* if its only minimum disconnecting sets are those induced by the neighbors of a vertex; a graph is said to be *tightly superconnected* if (i) any minimum disconnecting set is the set of neighbors of a single vertex; and (ii) the deletion of a minimum disconnecting set results in a graph with two components (one of which has only one vertex, another component is a connected graph). A $(\delta; g) - cage$ is a δ-regular graph with girth g and with the least possible number of vertices. In this paper we consider the problem of whether or not $(4; g)$-cages for $g \geq 5$ are tightly superconnected. We present some partial results and the remaining open problems.

Keywords: cages, maximally connected, superconnected, tightly superconnected.

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1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered.

The vertex set of a graph G is denoted by $V(G)$. The set of vertices adjacent to a vertex v is denoted by $N(v)$. The *degree* of a vertex v is $deg(v) = |N(v)|$, and a graph is called *regular* when all the vertices have the same degree. The degree of a vertex v in an induced subgraph H of G is $deg_H(v) = |N(v) \cap V(H)|$. The *distance* $d(u, v)$ of two vertices u and v in $V(G)$ is the length of the shortest path between u and v. We also use the notion of a distance between a vertex w and a set of vertices $S \subset V$ defined as $d(w, S) = d_G(w, S) = \min\{d(w, s) : s \in S\}$. For every $v \in V$ and every positive integer $r \geq 0$, $N_r(v) = \{w \in V : d(w, v) = r\}$ denotes the *neighborhood of v at distance r*. Similarly, for $S \subset V$, the neighborhood of S at distance r is denoted $N_r(S) = \{w \in V : d(w, S) = r\}$. Observe that $N_0(S) = S$. When $r = 1$ we write $N(v)$ and $N(S)$, instead of $N_1(v)$ and $N_1(S)$. The *diameter* of a graph G, written as *D(G)*, is the maximum distance of any two vertices among all the vertices of G. A *matching* in G is a set of pairwise non-adjacent edges. A *perfect matching* is a matching which covers all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.

A graph G is *connected* if there is a path between any two vertices of G. If $X \subset V$ and $G - X$ is not connected, then X is said to be a *cutset*. Suppose that G is a connected graph. We say that G is *r-connected* if the deletion of at least t vertices of G is required to disconnect the graph. A noncomplete graph G is r-connected if deleting any set of fewer than r vertices results in a connected graph. A complete graph with $r+1$ vertices is r-connected. A graph with minimum degree δ is maximally connected if it is δ-connected. The notion of superconnectedness was first introduced in [5]. A graph is superconnected if its only minimum disconnecting sets are those induced by the neighbors of a vertex; that is, a minimum disconnecting set is the set of neighbors of a single vertex. (This is a much stronger property than requiring the connectivity to be δ.) If, in addition, the deletion of a minimum disconnecting set results in a graph with two components (one of which has only one vertex), then the graph is said to be *tightly superconnected*. Note that a graph can be loosely superconnected but not tightly superconnected: for example, the complete bipartite graph $K_{r,r}$ with $r \geq 3$ is superconnected but not tighttly superconnected.

The notion of tightly superconnected is equivalent to the notion of quasi 4-connected for the particular case of 3-regular graphs. A cubic graph G is said to be 4-connected if it is superconnected and the graph resulting from G by removing a vertex and its 3 neighbors is still connected.

Cages were introduced by Tutte [17]. The length of a shortest cycle in a graph G is called the *girth* of G. A k-regular graph with girth g is called a $(k;g)$*-graph*. A $(k;g)$-graph is called a $(k;g)$*-cage* if it has the least possible number of vertices. For example, the cycle C_g is the unique $(2;g)$-cage, the complete graph K_{k+1} is the unique $(k;3)$-cage, the complete bipartite graph $K_{k,k}$ is the $(k;4)$-cage. Daven and Rodgers [6] and Jiang and Mubayi [11] have proved that every $(k;g)$-cage with $k \geq 3$ is 3-connected. Recently, Marcote *et al.* showed that $(k;g)$-cages with $g \geq 10$ are 4-connected [14]. Furthermore, they showed that $(k;g)$-cages with girth $g = 6, 8$ are k-connected and that most cages with $g = 5$ are also k-connected [13]. In [12], Lin *et al.* proved the following theorem:

Theorem 1.1 [12] *Let G be a $(k;g)$-cage with $k \geq 3$ and odd girth $g \geq 7$. Then G is r-connected with $r \geq \sqrt{k+1}$.*

Furthermore, the following known results on cages are used for proving our results.

Theorem 1.2 [11] *Let G be a $(k;g)$-cage and S a cutset of G. Then the diameter of the induced subgraph $G[S]$ is not less than $\lfloor g/2 \rfloor$. Furthermore, the inequality is strict if $d_{G[S]}(u,v)$ is maximized for exactly one pair of vertices.*

Theorem 1.3 [7] *Let $\delta \geq 2$ and $g \geq 3$ be integers, and let $f(\delta,g)$ be the order of a (δ,g)-cage. The following statements hold:*

(i) $f(\delta,g) < f(\delta,g+1)$.

(ii) The diameter of a (δ,g)-cage is $D \leq g$.

We shall often use the following result concerning the existence of a vertex having distance to a cutset at least $\lfloor (g-1)/2 \rfloor$, where g is the girth.

Lemma 1.1 [8] *Let $G = (V,E)$ be a graph with minimum degree δ and girth g. Assume that $X \subset V$ is a minimum cutset $\kappa = |X| \leq \delta - 1$. Then, for any connected component C of $G - X$, there exists some vertex $u \in V(C)$ such that $d(u,X) \geq \lfloor (g-1)/2 \rfloor$.*

A cutset X of G is called a *non-trivial cutset* if X does not contain the neighborhood $N(u)$ of any vertex $u \notin X$. Provided that some non-trivial cutset exists, the *superconnectivity* of G denoted by κ_1 was defined in [1, 9] as:

$$\kappa_1 = \kappa_1(G) = \min\{|X| : X \text{ is a non-trivial cutset}\}.$$

A non-trivial cutset X is called a κ_1*-cut* if $|X| = \kappa_1$. Notice that if $\kappa_1 \leq \delta$, then $\kappa_1 = \kappa$ and that $\kappa_1 > \delta$ is a sufficient and necessary condition for G to be super-κ, since all the minimum disconnecting sets with cardinality equal to δ must be trivial. A *non-trivial edge-cut*, the *edge-superconnectivity* $\lambda_1 = \lambda_1(G)$ and a λ_1*-cut* are defined analogously.
Some known sufficient conditions on the diameter of a graph in terms of its girth to guarantee lower bounds on κ, λ, κ_1 and λ_1 are listed in the following theorem.

Theorem 1.4 *Let G be a graph with minimum degree $\delta \geq 2$, diameter D, girth g, edge minimum degree ξ, connectivities λ and κ and superconnectivities κ_1 and λ_1. Then,*

(i) [16] $\lambda = \delta$ if $D \leq 2\lfloor (g-1)/2 \rfloor$.

(ii) [16] $\kappa = \delta$ if $D \leq 2\lfloor (g-1)/2 \rfloor - 1$.

(iii) [3] $\lambda_1 = \xi$ if $D \leq g - 2$.

(iv) [2] $\kappa_1 \geq \xi$ if $D \leq g - 3$.

The following lemma is also very useful in our results.

Lemma 1.2 *[4] Let G be a κ_1-connected graph with odd girth and minimum degree $\delta \geq 3$. Let X be a κ_1-cut with $|X| = \delta$ and assume that there exists a connected component C of $G - X$ such that $\max\{d(u,X) : u \in V(C)\} = (g-3)/2$. Then the following assertions hold:*

(i) If $u \in V(C)$ is such that $d(u,X) = (g-3)/2$ and $|N_{(g-3)/2}(u) \cap X| = 1$, then u has degree $d(u) = \delta$ and $\delta - 1$ neighbors z such that $d(z,X) = (g-3)/2$ and $|N_{(g-3)/2}(z) \cap X| = 1$.

(ii) If $u \in V(C)$ is such that $d(u,X) = (g-3)/2$ and $|N_{(g-3)/2}(u) \cap X| = 1$, then $|N_{(g-1)/2}(u) \cap X| = \delta - 1$.

(iii) There exists a $(\delta-1)$-regular subgraph Γ such that for every vertex $w \in V(\Gamma)$, $d_G(w) = \delta$ and $d(w,X) = (g-3)/2$.

(iv) If $g = 5$ then $|N(X) \cap V(C)| \geq \delta(\delta-1)$. And if $g \geq 7$ then $|N(X) \cap V(C)| \geq (\delta-1)^2 + 2$.

The aim of this work is to prove that $(4;g)$-cages are tightly superconnected. Marcote *et al.* [15] proved that cubic cages are quasi 4-connected, i.e, cubic cages are tightly superconnected. Xu, Wang and Wang [18] showed that all $(4;g)$-cages are 4-connected. Therefore, our task consists of proving the following two points.

(i) A minimum disconnecting set of any $(4;g)$-cage is the set of the four neighbors of a single vertex, in other words, all $(4;g)$-cages with girth $g \geq 5$ are superconnected.

(ii) The deletion of a vertex and its four neighbors from a $(4;g)$-cage results in a connected graph.

2 Results

Theorem 2.1 *Every $(4;g)$-cage with odd girth $g \geq 5$ has $\lambda_1 = 6$.*

Proof. Open problem. ■

Proposition 2.1 *Let G be $(4;g)$-cage with odd girth $g \geq 5$. Assume that there exists a κ_1-cut X such that $|X| = \kappa_1 \leq 4$, and let C be a connected component of $G - X$. Then there exists a vertex $u \in V(C)$ such that $d(u,X) \geq (g-1)/2$.*

Proof. If $\kappa_1 \leq 3$, then the result of Lemma 1.2 is true by Lemma 1.1. Now let $\kappa_1 = |X| = 4$. Assume $\max\{d(u,X) : u \in V(C)\} = \mu \leq (g-1)/2 - 1 = (g-3)/2$, we reason by contradiction. By Lemma 1.2 (iv), $|N(X) \cap V(C)| \geq \delta^2 - \delta$ when $g = 5$ and $|N(X) \cap V(C)| \geq 3^2 + 2$ when $g \geq 7$. Since G is a regular graph $|N(X) \cap V(C')| \leq |N(X)| - |N(X) \cap V(C)| \leq 16 - (16-4) = 4$ when $g = 5$, and $|N(X) \cap V(C')| \leq 16 - (9+2) \leq 5$ when $g \geq 7$ (see Figure 1).

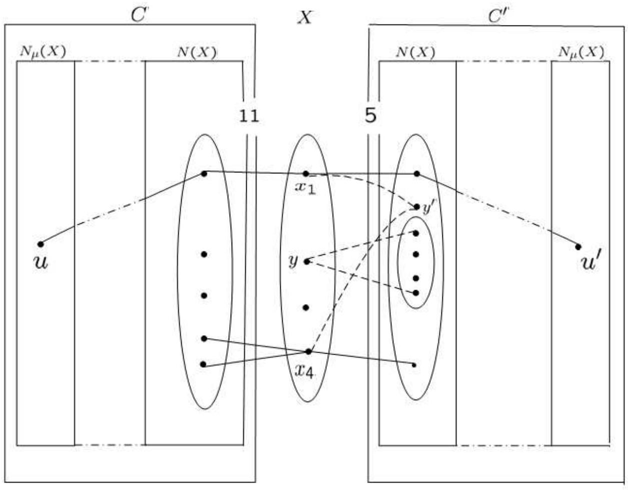

Figure 1: Illustration of the construction with $g \geq 7$ in the Theorem 1.2.

Let C' denote another component of $G - X$. Let $F' = [X, C']$ denote the set of edges having one vertex in X and the other vertex in C'. Then F' is an edge-cut such that $|F'| \leq 5$. Moveover F' is a nontrivial edge-cut, otherwise F' contains the edges incident with some vertex w in $X \cup C'$ (see Figure 1). If $w \in V(C')$ then X is a minimum trivial cutset which is a contradiction. Therefore $w \in X$ implying that $X - w$ is a cutset disconnecting C from $C' \cup \{w\}$, which contradicts X is a minimum nontrivial cutset. We conclude that F' is a nontrivial edge-cut with $|F'| < 6$. This contradicts Theorem 2.1, therefore $\mu \geq (g-1)/2$ and the theorem holds. ■

To prove that all $(4;g)$-cages with girth $g \geq 5$ are superconnected, we reason by contradiction assuming that there exists a κ_1-cut S such that $|S| = 4$. Let G_1 denote a smallest component of $G - S$. Thus we have $|V(G_1)| \leq |V(G_2)|$, where $G_2 = G - S - G_1$. Then we can write:

$$|V(G)| = |V(G_1)| + |S| + |V(G_2)| \geq 2|V(G_1)| + |S|. \tag{1}$$

By Proposition 2.1 there exists a vertex u having maximum distance $d(u,S) \geq (g-1)/2$. Then we consider two subgraphs,

$$N_1 = G[(V(G_1) - u - N(u)) \cup S],$$

$$N_2 = G[(V(G_1) - u) \cup S^*],$$

where S^* is a copy of S.
We will construct a 4-regular graph with girth at least g by using subgraphs N_1 and N_2. The order of the resulting graph will be

$$|N_1| + |N_2| < 2|G_1| + |S| \leq |V(G)|, \tag{2}$$

the strict inequality being due to (1). Thus we get a $(4;g)$-graph with fewer vertices than the number of vertices of the original graph G, and since G was assumed to be a $(4;g)$-cage, we obtain a contradiction with Theorem 1.3 (i).

To arrive at a contradiction, we consider separately two cases for the degree distribution of vertices s_i, where $s_i \in S$ and $i = 1,2,3,4$,

1. $deg_{G_1}(s_i) = 1$ or 3.
2. $deg_{G_1}(s_i) = 2$.

Theorem 2.2 *If $deg_{G_1}(s_i) = 1$ and $deg_{G_2}(s_i) = 3$ (respectively, $deg_{G_1}(s_i) = 3$ and $deg_{G_2}(s_i) = 1$), where $s_i \in S$ and $i = 1,2,3,4$, then G is not a $(4;g)$-cage.*

Proof. First, let us define two sets:

$W = \{w : w \in (\bigcup_{i=1}^4 N(u_i) \setminus u) \subset N_1, u_i \in N(u)\}$.
$Z = \{z : z \in N(u) \subset N_2\}$.
Additionally, $deg_{G_1}(s_i) = 3$ and $deg_{G_1}(s_i^*) = 1$ where $s_i \in S$, $s_i^* \in S^*$. We must add the necessary edges between N_1 and N_2 in order to get regularity. The idea is described below.

(a) There will be a perfect matching between every vertex w in W and an edge of s_i^* in G_2.

(b) There exists a one-to-one map between S and Z under the condition, that s_0 is not to be connected to z^*, such that $s_0 \in S$, and s_0 is the end vertex lying on the shortest path between u and S. Also $z^* \in Z$ and z^* is a vertex lying on the unique shortest path between u and S^*. The reason for uniqueness of the shortest path is that otherwise, a small cycle is formed by edges uz^*, uz, (z,t)-path and (z^*,t)-path where $z \in Z \backslash z^*$ and an unique vertex t, such that $t \in N(s_0^*) \cap G_1$, of total length $(g-3)/2 + (g-3)/2 = g-3$, which is impossible.

Thus we have constructed a new graph that is regular of degree 4 (see Figure 2).
Next we will show that the new graph has girth at least $g+1$.

First, let the girth g be odd. Consider two new edges ws_0^* and s_0z, where w is lying on the shortest path between u and S and s_0 is the end vertex of this (u,S)-path. Additionally, z is not lying on the shortest path between u and S^* and s_0^* is the end vertex of this (u,S^*)-path. It is obvious that the length of the shortest (w,s_0)-path is at least $(g-5)/2$. Therefore, the length of the shortest (z,s_0^*)-path is at least $(g+3)/2$, otherwise consider one vertex $z^* \in Z$, such that z^* is lying on the shortest (u,S^*)-path, and a unique vertex t, such that $t \in N(s_0^*) \cap G_1$. Due to $d(z^*,t) \geq (g-5)/2$, then $d(z,t) \geq g - ((g-5)/2 + 2) = (g+1)/2$. Then, $d(z,s_0^*) \geq d(z,t) + 1 = (g+3)/2$. Therefore, a cycle is formed by edges ws_0^*, s_0z, (w,s_0)-path and (z,s_0^*)-path, of total length $1+1+(g-5)/2+(g+3)/2 \geq g+1$, which is a contradiction.

On the other hand, let the girth g be even. We still consider these two new edges ws_0^* and s_0z, which have been described in the case of odd girth. It is obvious that the length of shortest (w,s_0)-path is at least $g/2 - 3$, and the length of shortest (z,s_0^*)-path is at least $g/2 + 2$, because $d(z^*,t) \geq g/2 - 3$, and then $d(z,s_0^*) \geq d(z,t) + 1 \geq (g - ((g/2 - 3) + 2)) + 1 = g/2 + 2$. Therefore, a cycle is formed by edges ws_0^*, s_0z, (w,s_0)-path and (z,s_0^*)-path, of total length

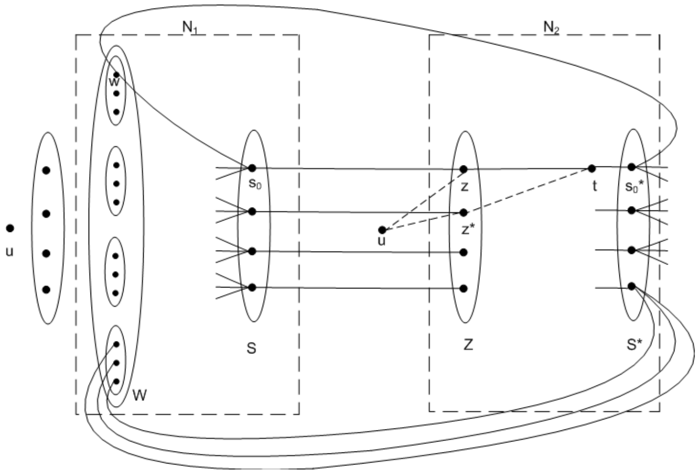

Figure 2: Illustration of the construction in the Lemma 2.2.

$1 + 1 + (g/2 - 3) + (g/2 + 2) \geq g + 1$, which is also a contradiction. ■

The next corollary is based on Theorem 1.2.

Corollary 2.1 *Let G be a $(k; g)$-cage with $k \geq 3$ and girth $g \geq 5$. Then the removal of any vertex $v \in V(G)$ and any subset of vertices adjacent to v results in a connected graphs.*

Proof. A *star cutset* is a cutset that contains a vertex adjacent to all the other vertices in the cutset. Since a star cutset induces a subgraph of diameter at most 2, a $(k; g)$-cage with odd girth contains a star cutset if the diameter $D(G[S]) \leq 2$. By applying Theorem 1.2, we obtain $(g-1)/2 \leq D(G[S]) \leq 2$. In other words, a $(k; g)$-cage contains no star cutset if $g \geq 5$. Therefore, if $k \geq 3$, $g \geq 5$ and G is a $(k; g)$-cage, then the deletion of a vertex and its all neighbors results in a connected graph. ■

3 Further Work

Proposition 2.1 is based on Theorem 2.1 which is an open problem. In order to prove it we would like to propose the following conjecture.

Conjecture 3.1 *Every $(k; g)$-cage with odd girth $g \geq 5$ has $\lambda_1 = 2k - 2$.*

Due to the fact that the Case 2 for the degree distribution of vertices s_i, where $s_i \in S$ and $i = 1, 2, 3, 4$, is still open, we would also like to propose the following open problem:

Open Problem 3.1 *Let G be a connected 4-regular graph with odd girth $g \geq 5$. Let S be a κ_1-cut of $|S| = 4$, and let G_1 be the smallest connected component of $G - S$. Thus we have $|V(G_1)| \leq |V(G_2)|$, where $G_2 = G - S - G_1$. Is it true that if $deg_{G_1}(s_i) = 2$ and $deg_{G_2}(s_i) = 2$, where $s_i \in S$ and $i = 1, 2, 3, 4$, then G is not a $(4; g)$-cage?*

Furthermore, since we believe that the answer to the above open problem is 'yes', we propose the following conjecture.

Conjecture 3.2 *Every $(4; g)$-cage, for odd $g \geq 5$, is tightly superconnected.*

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