Rectangle-of-Influence Drawings of Four-Connected Plane Graphs (Extended Abstract)

Kazuyuki Miura[1] **Takao Nishizeki**[2]

Graduate School of Information Sciences
Tohoku University, Aoba-yama 05, Sendai 980-8579, Japan
E-mail: [1]miura@nishizeki.ecei.tohoku.ac.jp
[2]nishi@ecei.tohoku.ac.jp

Abstract

A rectangle-of-influence drawing of a plane graph G is a straight-line planar drawing of G such that there is no vertex in the proper inside of the axis-parallel rectangle defined by the two ends of any edge. In this paper, we show that any 4-connected plane graph G with four or more vertices on the outer face has a rectangle-of-influence drawing in an integer grid such that $W + H \leq n$, where n is the number of vertices in G, W is the width and H is the height of the grid. Thus the area $W \times H$ of the grid is at most $\lceil (n-1)/2 \rceil \cdot \lfloor (n-1)/2 \rfloor$. Our bounds on the grid sizes are optimal in a sense that there exist an infinite number of 4-connected plane graphs whose drawings need grids such that $W + H = n - 1$ and $W \times H = \lceil (n-1)/2 \rceil \cdot \lfloor (n-1)/2 \rfloor$. We also show that the drawing can be found in linear time.

1 Introduction

Recently automatic aesthetic drawing of graphs have created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out (Battista, Eades, Tamassia & Tollis 1999, Chrobak & Kant 1997, Chiba, Onoguchi & Nishizeki 1985, Chrobak & Payne 1995, Chiba, Yamanouchi & Nishizeki 1984, Fáry 1984, de Fraysseix, Pach & Pollack 1990, He 1997, Kant 1996, Schnyder 1990, Tutte 1963). In this paper, we deal with the "rectangle-of-influence drawing" of a plane graph (Biedl, Bretscher & Meijer 1999). Throughout the paper we denote by n the number of vertices of a graph G. The $W \times H$ *integer grid* consists of $W + 1$ vertical grid lines and $H + 1$ horizontal grid lines, and has a rectangular contour. W and H are called the *width* and *height* of the integer grid, respectively.

The most typical drawing of a plane graph G is the *straight-line drawing* in which all vertices of G are drawn as points and all edges are drawn as straight line segments without any edge-intersection. A straight-line drawing of G is called a *grid drawing* of G if all vertices of G are put on grid points of integer coordinates. Every plane graph has a grid drawing on an $(n-2) \times (n-2)$ grid if $n \geq 3$ (Battista, Eades, Tamassia & Tollis 1999, Chrobak & Payne 1995, de Fraysseix, Pach & Pollack 1990, Schnyder 1990).

There are many results on straight-line drawings under additional constraints. For example, a straight-line drawing of a plane graph G is often pretty if every face boundary is drawn as a convex polygon (Chiba, Onoguchi & Nishizeki 1985, Tutte 1963). Such a drawing is called a *convex drawing* of G. Not every plane graph has a convex drawing, but every 3-connected plane graph has a convex drawing (Tutte 1963), and such a convex drawing can be found in linear time (Chiba, Onoguchi & Nishizeki 1985, Chiba, Yamanouchi & Nishizeki 1984). A convex drawing is called a *convex grid drawing* if it is a grid drawing. Every 3-connected plane graph has a convex grid drawing on an $(n-2) \times (n-2)$ grid, and such a grid drawing can be found in linear time (Chrobak & Kant 1997, Schnyder & Trotter 1992). Figure 1(a) depicts a convex grid drawing of a plane graph obtained by the algorithm in (Chrobak & Kant 1997). On the other hand, a restricted class of graphs has a more compact convex grid drawing. For example, if G is a 4-connected plane graph and has at least four vertices on its outer face, then G has a convex grid drawing on a $W \times H$ grid such that $W + H \leq n - 1$, and one can find such a convex grid drawing in linear time (Miura, Nakano & Nishizeki 2000). Figure 1(b) depicts a convex grid drawing of the same graph obtained by the algorithm in (Miura, Nakano & Nishizeki 2000).

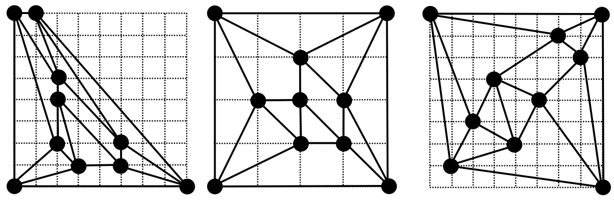

Figure 1: (a) Convex grid drawing, (b) open rectangle-of-influence drawing which is a convex drawing, and (c) closed rectangle-of-influence drawing of a plane graph.

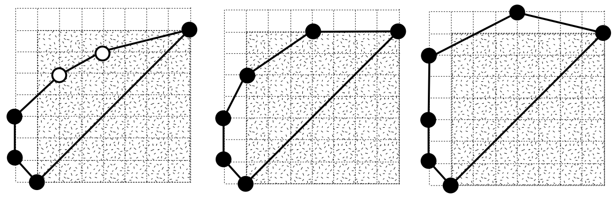

Figure 2: A face of (a) a convex grid drawing, (b) an open rectangle-of-influence drawing, and (c) a closed rectangle-of-influence drawing.

In this paper, we deal with a type of straight-line drawings under another additional constraint, known as the *(open) rectangle-of-influence drawing*; it is a straight-line drawing such that there is no vertex in the proper inside of the axis-parallel rectangle defined by the two ends of any edge. A rectangle-of-influence drawing often looks pretty, since vertices are inclined

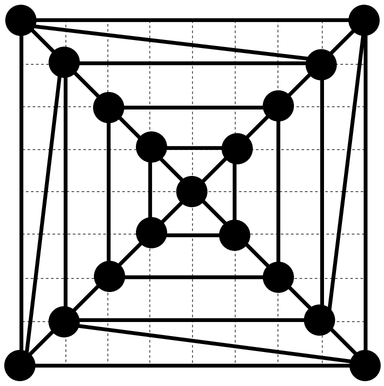

Figure 3: Nested quadrangles attaining our bounds.

to be separated from edges. The convex drawing in Fig. 1(a) is not a rectangle-of-influence drawing, while the convex drawing in Fig. 1(b) is a rectangle-of-influence drawing. A rectangle-of-influence drawing is called *closed* if the axis-parallel rectangle defined by the two ends of any edge contains no vertices except the ends on its boundary. Figure. 1(c) depicts a closed rectangle-of-influence drawing of the same plane graph as in Figs. 1(a) and (b). Figures 2(a), (b) and (c) depict a face of a convex drawing, an open rectangle-of-influence drawing, and a closed rectangle-of-influence drawing, respectively, where the axis-parallel rectangle defined by the ends of an edge is shaded.

Biedl *et al.* showed that a plane graph G has a (closed) rectangle-of-influence drawing on an $(n-1)\times(n-1)$ grid if there is no vertices in the interior of an any 3-cycle (Biedl, Bretscher & Meijer 1999). The closed rectangle-of-influence drawing in Fig. 1(c) is obtained by their algorithm. It is not a convex drawing. Their result implies that any 4-connected plane graph with four or more vertices on the outer face has a (closed) rectangle-of-influence drawing on an $(n-1)\times(n-1)$ grid. However, the size of an integer grid required by a rectangle-of-influence drawing would be smaller than $(n-1)\times(n-1)$ for 4-connected plane graphs, but it has not been known how small the grid size is.

In this paper we give an answer to this problem. That is, we show that the convex grid drawing of a 4-connected plane graph G found by the algorithm of Miura *et al.* (Miura, Nakano & Nishizeki 2000) is always an (open) rectangle-of-influence drawing of G, and hence one can find in linear time a rectangle-of-influence grid drawing of G on a $W \times H$ grid such that $W + H \leq n-1$ if G has n vertices. Since $W + H \leq n - 1$, the area $W \times H$ satisfies $W \times H \leq \lceil (n-1)/2 \rceil \cdot \lfloor (n-1)/2 \rfloor$. The outer face boundary of G is always drawn as a rectangle as illustrated in Figs. 3 and 5(d). The assumption that a given plane graph has four or more vertices on the outer face does not lose the generality, because any 4-connected plane graph with exactly three vertices on the outer face has no rectangle-of-influence drawing. Our bounds on $W + H$ and $W \times H$ are optimal in a sense that there exist an infinite number of 4-connected plane graphs, for example the nested quadrangles depicted in Fig. 3, any drawings of which need grids such that $W+H = n-1$ and $W\times H = \lceil (n-1)/2\rceil \cdot \lfloor (n-1)/2 \rfloor$. It should be noted that any 4-connected plane graph with four or more vertices on the outer face has a closed (and hence open) rectangle-of-influence drawing on an $(n-1)\times(n-1)$ grid (Biedl, Bretscher & Meijer 1999), but the drawing is not always a convex drawing as illustrated in Fig. 1(c).

The remainder of the paper is organized as follows. In Section 2, we give some definitions. In Section 3, we outline the linear-time algorithm in (Miura, Nakano & Nishizeki 2000) for finding a convex drawing of a 4-connected plane graph. In Section 4, we prove that the convex drawing is an open rectangle-of-influence drawing.

2 Preliminaries

In this section we introduce some definitions.

Let $G = (V, E)$ be a simple connected undirected graph having no multiple edge or loop. V is the vertex set, and E is the edge set of G. Let $x(v)$ and $y(v)$ be the x- and y-coordinates of vertex $v \in V$, respectively. An edge joining vertices u and v is denoted by (u, v). The *degree* of a vertex v in G is the number of neighbors of v in G, and is denoted by $d(v, G)$.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph divides the plane into connected regions called *faces*. We denote the boundary of a face by a clockwise sequence of the vertices on the boundary. We call the boundary of the outer face of a plane graph G *the contour* of G, and denote it by $C_o(G)$.

The *open rectangle of an edge* is defined to be the interior of the rectangle defined by the ends of the edge. *An (open) rectangle-of-influence drawing of* G is a straight-line planar drawing of G such that there are no vertices in the open rectangle of any edge.

The “4-canonical decomposition” of a plane graph $G = (V, E)$ (Nakano, Rahman & Nishizeki 1997) playing a crucial role in the algorithm in (Miura, Nakano & Nishizeki 2000) is a generalization of two well-known concepts: the “canonical ordering,” which is used to find a convex grid drawing of a 3-connected plane graph (Kant 1996); and the “4-canonical ordering,” which is used to find a “visibility representation” and a grid drawing of a 4-connected plane graph (He 1997, Kant & He 1997, Miura, Nakano & Nishizeki 1999). A 4-canonical decomposition $\Pi = (U_1, U_2, \cdots, U_{12})$ is illustrated in Fig. 4 for a 4-connected plane graph. Let m be a natural number, and let $\Pi = (U_1, U_2, \cdots, U_m)$ be a partition of set V to m subsets $U_1, U_2, \cdots, U_m$ of V where $U_1 \bigcup U_2 \bigcup \cdots \bigcup U_m = V$ and $U_i \bigcap U_j = \phi$ for any i and j, $i \neq j$. Let G_k, $1 \leq k \leq m$, be the plane subgraph of G induced by the vertices in $U_1 \bigcup U_2 \bigcup \cdots \bigcup U_k$, and let $\overline{G_k}$ be the plane subgraph of G induced by the vertices in $U_{k+1} \bigcup U_{k+2} \bigcup \cdots \bigcup U_m$. Thus $G = G_m = \overline{G_0}$. We say that Π is a *4-canonical decomposition* of G if the following three conditions are satisfied:

(co1) U_1 consists of the two ends of an edge on $C_o(G)$, and U_m consists of the two ends of another edge on $C_o(G)$;

(co2) for each k, $2 \leq k \leq m-1$, both G_k and $\overline{G_{k-1}}$ are biconnected and

(co3) for each k, $2 \leq k \leq m-1$, one of the following three conditions holds:

(a) U_k is a singleton set of a vertex u on $C_o(G_k)$ such that $d(u, G_k) \geq 2$ and $d(u, \overline{G_{k-1}}) \geq 2$.

(b) U_k is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) = 2$ and $d(u, \overline{G_{k-1}}) \geq 3$ for each vertex $u \in U_k$.

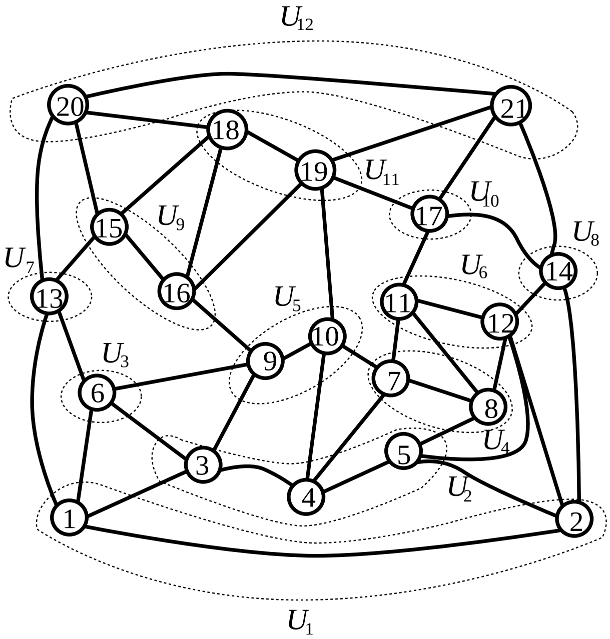

Figure 4: A 4-canonical decomposition of a 4-connected plane graph having $n = 21$ vertices.

(c) U_k is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) \geq 3$ and $d(u, \overline{G_{k-1}}) = 2$ for each vertex $u \in U_k$.

By the condition (co3), one may assume that for each k, $1 \leq k \leq m$, the vertices in U_k consecutively appear clockwise on $C_o(G_k)$. However, the clockwise order on $C_o(G_1)$ is not well-defined since $G_1 = K_2$. So we assume that the two vertices in U_1 consecutively appear counterclockwise on $C_o(G)$ as illustrated in Fig. 4. We number all vertices of G by $1, 2, \cdots, n$ so that they appear in $U_1, U_2, \cdots, U_m$ in this order, and call each vertex in G by the number i, $1 \leq i \leq n$. Thus one can define an order $<$ among the vertices in G.

In the remainder of this section, we define some terms which are used in the algorithm in (Miura, Nakano & Nishizeki 2000). The *lower neighbor* of u is the neighbors of u which are smaller than u. The *upper neighbor* of u is the neighbors of u which are larger than u. Every upper neighbor v of any vertex u satisfies $y(v) \geq y(u)$ in our drawing. The number of lower neighbors of u is denoted by $d_{low}(u, G)$, and the number of upper neighbors of u is denoted by $d_{up}(u, G)$. Every vertex u except vertex 1 satisfies $d_{low}(u, G) \geq 1$, and every vertex u except vertex n satisfies $d_{up}(u, G) \geq 1$. Let $2 \leq k \leq m - 1$ and $U_k = \{u_1, u_2, \cdots, u_h\}$. If U_k satisfies the condition (co3)(a), then $h = 1$ and $d_{low}(u_1), d_{up}(u_1) \geq 2$. If U_k satisfies condition (co3)(b), then $d_{low}(u_i) = 1$ for each u_i, $1 \leq i \leq h-1$, $d_{low}(u_h) = 2$, $d_{up}(u_i) \geq 3$ for each u_i, $1 \leq i \leq h-1$, and $d_{up}(u_h) \geq 2$. If U_k satisfies condition (co3)(c), then $d_{low}(u_1) \geq 2$, $d_{low}(u_i) \geq 3$ for each u_i, $2 \leq i \leq h$, $d_{up}(u_1) = 2$, and $d_{up}(u_i) = 1$ for each u_i, $2 \leq i \leq h$. We denote by $w_m(u)$ the largest neighbor of u, $1 \leq u \leq n - 1$. The *in-degree* of a vertex u in a directed graph D is denoted by $d_{in}(u, D)$, while the *out-degree* of u is denoted by $d_{out}(u, D)$.

3 Algorithm

In this section, we outline the algorithm in (Miura, Nakano & Nishizeki 2000). The algorithm first decides the x-coordinates of all vertices, and then decide the y-coordinates.

3.1 How to compute x-coordinates

The following **procedure** Construct-F constructs a directed forest $F = (V, E_F)$. All vertices in each component of F have the same x-coordinate; if there is a directed edge (i, j) in F, then $x(j) = x(i)$ and $y(j) > y(i)$.

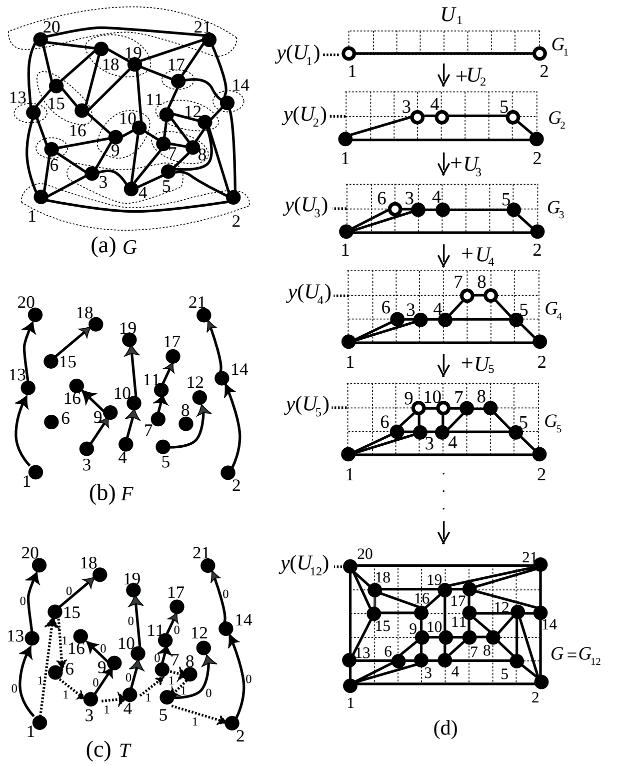

Figure 5: Illustration of algorithm.

Procedure Construct-F
begin $\{F = (V, E_F)\}$
1 $E_F := \phi$; {the initial forest $F = (V, \phi)$ consists of isolated vertices}
2 **for** $i := 1$ **to** n **do**
  if vertex i has upper neighbors j such that $d_{in}(j, F) = 0$ **then**
3   let j be the largest one among them, and add a directed edge (i, j) to the directed graph F, that is, $E_F := E_F \bigcup \{(i, j)\}$;
end.

Fig. 5(b) illustrates the directed forest F of the graph G in Fig. 5(a). Both the path $1, 13, 20$ going clockwise on $C_o(G)$ from 1 to $n - 1 = 20$ and the path $2, 14, 21$ going counterclockwise on $C_o(G)$ from 2 to $n = 21$ are directed paths in F, and hence these two paths are put on vertical grid lines as shown in the bottom figure of Fig. 5(d). Each of the other paths in F is put on a vertical grid line, too.

We then show how to arrange the paths in F from left to right. The algorithm decides a total order among all starting vertices of paths in F. For this purpose, using the following **procedure** Total-Order, the algorithm finds a directed path P going from vertex 1 to vertex 2 passing through all starting vertices of F. In Fig. 5(c), the directed path P is drawn by dotted lines.

Procedure Total-Order
begin
1 let P be the path directly going from vertex 1 to vertex 2;
2 **for** $i := 3$ **to** n **do**

if $d_{in}(i, F) = 0$ **then** $\{i$ is a starting vertex of a path in $F\}$
begin
let j be the first lower neighbor of i in the i's adjacency list in which the i's neighbors appear counterclockwise around i, and the first element of which is $w_m(i)$;
let j' be the starting vertex of the path in F containing vertex j; $\{2 \neq j' < i\}$
let k be the successor of j' in path P; {the path starting from vertex k in F has been put next to the right of the path starting from vertex j'}
insert i in P between j' and k; {the path starting from i in F is put between the path starting from j' and the path starting from k}
end
end.

The algorithm constructs a weighted tree T rooted at vertex 1 by adding the path P to the forest F; every edge of F has weight 0, and every edge of P has weight 1 in T. Then the x-coordinate $x(i)$ of each vertex i, $1 \leq i \leq n$, is the length of the path from root 1 to i in T. Thus $x(1) = 0$, and $W = x(2)$.

3.2 How to compute y-coordinates

We now show how to compute y-coordinates. For each k, $1 \leq k \leq m$, y-coordinates of all vertices in $U_k = \{u_1, u_2, \cdots, u_h\}$ are decided as the same integer, which is denoted by $y(U_k)$. Thus the path $u_1, u_2, \cdots, u_h$ on $C_o(G_k)$ is drawn as a horizontal line segment connecting points $(x(u_1), y(U_k))$ and $(x(u_h), y(U_k))$. (See Fig. 5(d).) Furthermore, the algorithm decides the y-coordinates $y(U_1), y(U_2), \cdots, y(U_m)$ in this order. Thus $H = y(U_m)$.

The algorithm first decides the y-coordinate $y(U_1)$ of $U_1 = \{1, 2\}$ as $y(U_k) = 0$. Thus it draws $G_1 = K_2$ as a horizontal line segment connecting points $(x(1), 0)$ and $(x(2), 0)$, as illustrated in the top figure of Fig. 5(d).

Suppose that $y(U_1), y(U_2), \cdots, y(U_{k-1})$, $k \geq 2$, have already been decided, that is, G_{k-1} has already been drawn, and one is now going to decide $y(U_k)$ and obtain a drawing of G_k by adding the vertices in U_k to the drawing of G_{k-1}. Let $C_o(G_{k-1}) = w_1, w_2, \cdots, w_t$, where $w_1 = 1$ and $w_t = 2$. Let $C_o(G_k) = w_1, w_2, \cdots, w_l, u_1, u_2, \cdots, u_h, w_r, \cdots, w_t$, where $1 \leq l < r \leq t$. Let y_{max} be the maximum value of y-coordinates of vertices $w_l, w_{l+1}, \cdots, w_r$; all these vertices were on $C_o(G_{k-1})$, but all these vertices except w_l and w_r are not on $C_o(G_k)$. Clearly one must decide $y(U_k) \geq y_{max}$ to obtain a plane drawing of G_k. The algorithm decides $y(U_k)$ to be either y_{max} or $y_{max} + 1$ so that the height H of the drawing becomes as small as possible. There are the following six cases.

Case 1: $y_{max} > y(w_l), y(w_r)$. (See Fig. **??**(a).)
In this case, if one decided $y(U_k) = y_{max}$, then G_k could not be a plane drawing. Therefore the algorithm decides $y(U_k) = y_{max} + 1$.

Case 2: $y_{max} = y(w_l) = y(w_r)$.
In this case, if one decided $y(U_k) = y_{max}$, then G_k might not be a plane drawing. Therefore the algorithm decides $y(U_k) = y_{max} + 1$.

Case 3: $y_{max} = y(w_l) > y(w_r)$, and F has a directed edge (w_l, u_1), that is, $x(w_l) = x(u_1)$.
In this case, if one decided $y(U_k) = y_{max}$, then vertices w_l and u_1 would overlap each other. Therefore the algorithm decides $y(U_k) = y_{max} + 1$.

Case 4: $y_{max} = y(w_l) > y(w_r)$, and F does not have a directed edge (w_l, u_1), that is, $x(w_l) < x(u_1)$.
In this case, the algorithm decides $y(U_k) = y_{max}$.

Case 5: $y_{max} = y(w_r) > y(w_l)$, and F has a directed edge (w_r, u_h), that is, $x(w_r) = x(u_h)$.
In this case, if one decided $y(U_k) = y_{max}$, then vertices w_r and u_h would overlap each other. Therefore the algorithm decides $y(U_k) = y_{max} + 1$.

Case 6: $y_{max} = y(w_r) > y(w_l)$, and F does not have a directed edge (w_r, u_h), that is, $x(u_h) < x(w_r)$.
In this case, the algorithm decides $y(U_k) = y_{max}$.

4 Proof for open rectangle-of-influence drawing

In this section, we prove that the convex drawing of a 4-connected plane graph G found by the algorithm of Miura *et al.* (Miura, Nakano & Nishizeki 2000) is an open rectangle-of-influence drawing of G.

We first show that, each face of the convex drawing found by the algorithm of Miura *et al.* (Miura, Nakano & Nishizeki 2000) is a particular convex polygon called an "trimmed rectangle." We call a polygon a *trimmed rectangle* if it can be obtained from an axis-parallel rectangle by trimming off some of the four corners, as illustrated in Fig. 6. Thus it has at least three sides and at most eight sides, and at most two of them are horizontal, at most two are vertical and at most four are oblique. We will prove the following lemma later.

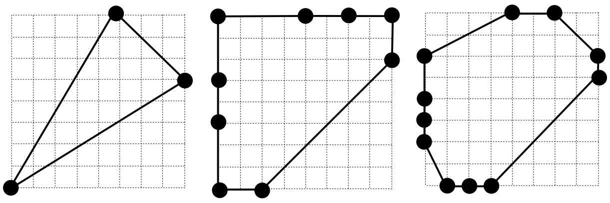

Figure 6: Trimmed rectangles.

Lemma 4.1 In the convex drawing of G found by the algorithm of Miura *et al.* (Miura, Nakano & Nishizeki 2000), every face boundary is drawn as a trimmed rectangle, and each of the oblique sides is exactly one edge of G as illustrated in Fig. 6.

We immediately have the following theorem from Lemma 4.1.

Theorem 1 The convex drawing found by the algorithm of Miura *et al.* (Miura, Nakano & Nishizeki 2000) is an open rectangle-of-influence drawing of G.

Proof If an edge is drawn as a horizontal or vertical line segment, then there is no vertex in the open rectangle of the edge. We shall thus show that there is no vertex in the open rectangle of any edge e drawn as an oblique line segment.

Assume for a contradiction that there is a vertex u in the open rectangle-of-influence of e. We may assume without loss of generality that e is drawn as a line segment of positive slope as illustrated in Fig. 7. The edge e divides the open rectangle of e into two right-angled triangles, the upper triangle and the lower one. One may assume that u is in the upper triangle.

We now claim that both edge e and vertex u are assumed to be on the boundary of the same face F, as illustrated in Fig. 7. Suppose conversely that the boundary of the upper face F containing edge e does

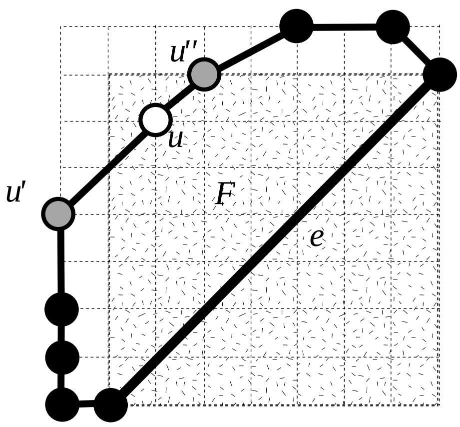

Figure 7: Illustration for the proof of Theorem 1.

not contain vertex u. Then by Lemma 4.1 F is a trimmed rectangle, having one or two oblique sides of positive slope, including e The face F has the other oblique side of positive slope; otherwise, the open rectangle of e would be contained in F and hence would not contain any vertex. Clearly, the open rectangle of $e^{'}$ contains u. If both $e^{'}$ and u are not on the contour of the same face, then one can find an edge $e^{''}$ such that $e^{''}$ is oblique side of a face and the open rectangle of $e^{''}$ contain u. Repeating these arguments, one can eventually find an edge e and a vertex u such that u is in the open rectangle of e and both e and u are on the same face.

We then derive a contradiction, as follows. Denote the boundary of the face F by $C_o(F)$. $C_o(F)$ contains e and u. Let $u^{'}$ be the preceding vertex of u on $C_o(F)$, and let $u^{''}$ be the succeeding vertex of u on $C_o(F)$. Since F is a convex polygon and u is in the open rectangle of e, we have $x(u^{'}) < x(u) < x(u^{''})$ and $y(u^{'}) < y(u) < y(u^{''})$. Thus both edges $(u^{'}, u)$ and $(u, u^{''})$ are drawn as positive line segments of positive slope. Hence F would have an oblique side of two or more edges, contrary to Lemma 4.1. Q.E.D

In the remainder of this section, we prove Lemma 4.1.

The algorithm in (Miura, Nakano & Nishizeki 2000) finds the drawing of $G_1, G_2, \cdots, G_m (= G)$ in this order, as illustrated in Fig. 5(d). Thus, assuming that each face of the drawing of G_{k-1}, $k \geq 2$, is drawn as a trimmed rectangle, we shall show that each face of the drawing of G_k is drawn as a trimmed rectangle. However, it is difficult to show that each face of the drawing of G_k is drawn as a trimmed rectangle for the case where either $k = m$ or U_k, $2 \leq k \leq m-1$, satisfies the condition (co3)(c). Therefore, subdividing all such sets U_k, we obtain another partition of V as follows. Let $\Pi = (U_1, U_2, \cdots, U_m)$ be a 4-canonical decomposition of G. For each U_k such that either $k = m$ or U_k satisfies the condition (co3)(c), let $U_k = \{u_1, u_2, \cdots, u_{l_k}\}$ and replace U_k in Π with singleton sets $\{u_1\}, \{u_2\}, \cdots, \{u_{l_k}\}$. We call the resulting partition $\Pi^{'} = (U_1, U_2^1, U_2^2, \cdots, U_2^{l_2}, U_3^1, U_3^2, \cdots, U_3^{l_3}, \cdots, U_m^1, U_m^2)$ of V a *refined decomposition* of G. If either $k = m$ or U_k satisfies the condition (co3)(c), then $U_k = U_k^1 \bigcup U_k^2 \bigcup \cdots \bigcup U_k^{l_k}$, $l_k = |U_k|$ and $|U_k^i| = 1$ for each i, $1 \leq i \leq l_k$. Otherwise, $l_k = 1$ and $U_k = U_k^1$.

For each k, $2 \leq k \leq m$, and for each i, $1 \leq i \leq l_k$, we denote by G_k^i the plane subgraph of G induced by the vertices in $U_1 \bigcup U_2 \bigcup \cdots \bigcup U_{k-1} \bigcup U_k^1 \bigcup U_k^2 \bigcup \cdots \bigcup U_k^i$. Moreover, for each k, $2 \leq k \leq m$, and for each i, $0 \leq i \leq l_k - 1$, we denote by $\overline{G_k^i}$ the plane subgraph of G induced by the vertices in $U_k^{i+1} \bigcup U_k^{i+2} \bigcup \cdots \bigcup U_k^{l_k} \bigcup U_{k+1} \bigcup \cdots \bigcup U_m$. For notational convenience, let $G_k^0 = G_{k-1}$ and $\overline{G_k^{l_k}} = \overline{G_k}$.

Let $k \geq 2$ and $U_k^i = \{u_1, u_2, \cdots, u_h\}$. By the definition of a refined decomposition, vertices $u_1, u_2, \cdots, u_h$ consecutively appear clockwise on $C_o(G_k^i)$ in this order. Let $C_o(G_k^{i-1}) = w_1, w_2, \cdots, w_t$, where $w_1 = 1$ and $w_t = 2$. Let $C_o(G_k^i) = w_1, w_2, \cdots, w_l, u_1, u_2, \cdots, u_h, w_r, \cdots, w_t$, where $1 \leq l < r \leq t$. We call w_l the *left leg* of U_k^i, and w_r the *right leg* of U_k^i. By the definition of a 4-canonical decomposition and a refined decomposition, the left leg of U_k^i is different from the right leg of U_k^i.

We now have the following lemma for the drawing of G_k^i.

Lemma 4.2 For each k, $2 \leq k \leq m$, and each i, $0 \leq i \leq l_k$, the following (i)–(iii) hold:

(i) the path going clockwise on $C_o(G_k^{i-1})$ from vertex $w_1 = 1$ to vertex $w_t = 2$ is "x-monotone," that is, $x(w_1) \leq x(w_2) \leq \cdots \leq x(w_t)$ (such a path is drawn by thick solid lines in Fig. 8);

(ii) the path going clockwise on $C_o(G_k^{i-1})$ from w_l to w_r is "quasi-convex," that is, there is no vertex w_p such that $l < p < r$ and $y(w_{p-1}) < y(w_p) > y(w_{p+1})$ (all vertices in such a path are drawn by gray circles in Fig. 8), and $w_l, w_{l+1}, \cdots, w_r, w_l$ is a convex polygon in particular if U_k satisfies the condition (co3)(b) (as illustrated in Fig. 8(a)); and

(iii) if a vertex v on $C_o(G_k^{i-1})$ is an inner vertex of G, that is, v is not on $C_o(G)$, and satisfies one of the following three conditions (a)-(c), then v has at least one neighbor in $\overline{G_k^{i-1}}$ (the edges joining v and such neighbors are drawn by thin dotted line in Fig. 8).

(a) v is a common end vertex of two edges consecutively appearing on $C_o(G_k^{i-1})$ having positive slopes.

(b) v is a common end vertex of two edges consecutively appearing on $C_o(G_k^{i-1})$ having negative slopes.

(c) The interior angle of the polygon $C_o(G_k^{i-1})$ at vertex v is less than 180°.

Proof Investigating the algorithm in detail, one can prove the lemma. The detail is omitted in this extended abstract. Q.E.D

Proof of Lemma 4.1 Using Lemma 4.2, one can prove Lemma 4.1. Note that all inner face boundaries newly formed in G_k^i are trimmed rectangles as illustrated in Fig. 8 (all such faces are not shaded in Fig. 8). Q.E.D

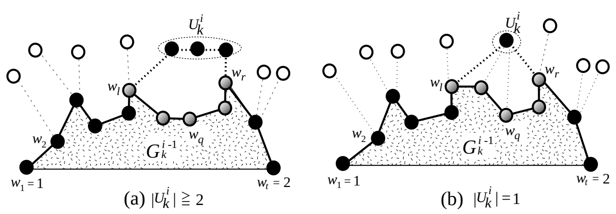

Figure 8: Illustration for Lemma 4.3.

5 Conclusion

In this paper we show that any 4-connected plane graph G has an open rectangle-of-influence drawing on an integer grid such that $W + H \le n - 1$ if G has four or more vertices on the outer face boundary, and one can find such a drawing in linear time. Since $W + H \le n - 1$, the area $W \times H$ of the grid is at most $\lceil (n-1)/2 \rceil \cdot \lfloor (n-1)/2 \rfloor$. Our bounds on $W + H$ and $W \times H$ are optimal in a sense that there exist an infinite number of 4-connected plane graphs which need grids such that $W + H = n + 1$ and $W \times H = \lceil (n-1)/2 \rceil \cdot \lfloor (n-1)/2 \rfloor$.

Acknowledgment

We thank Shin-ichi Nakano for valuable discussion.

References

T. C. Biedl, A. Bretscher & H. Meijer. (1999), Rectangle of influence drawings of graphs without filled 3-cycles, Proc. Graph Drawing'99 (GD99), LNCS 1731, pp 359–368.

G. Di Battista, P. Eades, R. Tamassia & I.G. Tollis. (1999), Graph Drawing, Prentice Hall, NJ.

M. Chrobak & G. Kant. (1997), Convex grid drawings of 3-connected planar graphs, International Journal of Computational Geometry and Applications, 7, pp. 211-223.

N. Chiba, K. Onoguchi & T. Nishizeki. (1985), Drawing planar graphs nicely, Acta Inform., 22, pp. 187–201.

M. Chrobak & T. Payne. (1995), A linear-time algorithm for drawing planar graphs on a grid, Information Processing Letters, 54, pp. 241–246.

N. Chiba, T. Yamanouchi & T. Nishizeki. (1984), Linear algorithms for convex drawings of planar graphs, in Progress in Graph Theory, J.A. Bondy and U.S.R. Murty (eds.), Academic Press, pp. 153–173.

I. Fáry. (1984), On straight lines representation of plane graphs, Acta. Sci. Math. Szeged, 11, pp. 229–233.

H. de Fraysseix, J. Pach & R. Pollack. (1990), How to draw a planar graph on a grid, Combinatorica, 10, pp41–51.

X. He. (1997), Grid embedding of 4-connected plane graphs, Discrete & Computational Geometry, 17, pp. 339–358.

G. Kant. (1996), Drawing planar graphs using the canonical ordering, Algorithmica, 16, pp. 4–32.

G. Kant & X. He. (1997), Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems, Theoretical Computer Science, 172, pp. 175–193.

K. Miura, S. Nakano & T. Nishizeki. (1999), Grid drawings of four-connected plane graphs, Proc. Graph Drawing'99 (GD'99), LNCS 1731, pp. 145–154.

K. Miura, S. Nakano & T. Nishizeki. (2000), Convex grid drawings of four-connected plane graphs, Proceedings of 11th Annual International Symposium on Algorithms and Computation (ISAAC'00), LNCS 1969, pp. 254–265.

S. Nakano, M. Saidur Rahman & T. Nishizeki. (1997), A linear time algorithm for four partitioning four-connected planar graphs, Information Processing Letters, 62, pp. 315–322.

W. Schnyder. (1990), Embedding planar graphs in the grid, Proc. 1st Annual ACM-SIAM Symp. on Discrete Algorithms, San Francisco, pp. 138–147.

W. Schnyder & W. Trotter. (1992), Convex drawings of planar graphs, Abstracts of the AMS, 13, 5, 92T-05-135.

W.T. Tutte. (1963), How to draw a graph, Proc. London Math. Soc., 13, pp. 743–768.