Minimum Cost Source Location Problem with Local 3-Vertex-Connectivity Requirements

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Abstract

Let $G = (V, E)$ be a simple undirected graph with a set V of vertices and a set E of edges. Each vertex $v \in V$ has a demand $d(v) \in Z_+$ and a cost $c(v) \in R_+$, where Z_+ and R_+ denote the set of nonnegative integers and the set of nonnegative reals, respectively. The source location problem with vertex-connectivity requirements in a given graph G asks to find a set S of vertices minimizing $\sum_{v \in S} c(v)$ such that there are at least $d(v)$ pairwise vertex-disjoint paths from S to v for each vertex $v \in V - S$. It is known that if there exists a vertex $v \in V$ with $d(v) \geq 4$, then the problem is NP-hard even in the case where every vertex has a uniform cost. In this paper, we show that the problem can be solved in $O(|V|^4(\log |V|)^2)$ time if $d(v) \leq 3$ holds for each vertex $v \in V$.

Keywords: graph algorithm, undirected graph, location problem, local vertex-connectivity, polynomial time algorithm

1 Introduction

Problems of selecting the best location of facilities in a given network to satisfy a certain property are called *location problems* (Labbe et al. 1995). Recently, the location problems with requirements measured by a network-connectivity were studied extensively (Arata et al. 2002, Bárász et al. 2004, Honami et al. 1999, Ishii et al. 2003, Ito et al. 2002, Ito et al. 2001, Ito et al. 2000, Ito & Yokoyama 1998, Nagamochi et al. 2001, Tamura et al. 1990, Tamura et al. 1998).

Connectivity and/or flow-amount are very important factors in applications to control and design of multimedia networks. In a multimedia network, some vertices of the network, such as the so-called mirror servers, may have functions of offering the same services for users. Let us call a vertex that can offer the service i a *source*, and let S be a set of sources, where we can locate more than one source in a network. A user at a vertex v can use the service i by communicating with at least one source $s \in S$ through a path between s and v. The flow-amount (which is the capacity of paths between S and v) affects the maximum data amount that can be transmitted from S to a user at a vertex v. Also, the edge-connectivity or the vertex-connectivity between a source set S and a vertex v measures the robustness of the service against network failures.

In this paper, we consider the problem of finding the best location of a source set S under connectivity and/or flow-amount requirements from each vertex to a source set S. We introduce the *source location problem* formulated as follows.

Problem 1.1 (*Source location problems*)
Input : A graph $G = (V, E)$ with a set V of vertices and a set E of edges capacitated by nonnegative reals, a cost function $c : V \to R_+$ (where R_+ denotes the set of nonnegative reals), and a demand function $d : V \to R_+$.
Output : A vertex set $S \subseteq V$ such that $\psi(S, v) \geq d(v)$ holds for every vertex $v \in V - S$ and $\sum_{v \in S} c(v)$ is the minimum, where $\psi(S, v)$ is a measurement based on the edge-connectivity, the vertex-connectivity or the flow-amount between S and a vertex v in a graph G. □

For such measurements $\psi(S, v)$, one may consider the minimum size $\lambda(S, v)$ of an edge cut $C \subseteq E$ that separates v from S, the minimum size $\kappa(S, v)$ of a vertex cut $C \subseteq V - S - v$ that separates S and v, or the maximum number $\hat{\kappa}(S, v)$ of vertex-disjoint paths between S and v such that no two paths meet at the same vertex in S.

Source location problems with $\psi = \lambda$ in undirected graphs were treated by Tamura et al. (Tamura et al. 1990, Tamura et al. 1998), Ito et al. (Ito et al. 2000, Ito & Yokoyama 1998) and Arata et al. (Arata et al. 2002). They gave polynomial time algorithms for uniform costs $c(v) = 1$, $v \in V$, while the problem with general costs $c(v)$, $v \in V$ is shown to be weakly NP-hard (Arata et al. 2002). In directed graphs, Ito et al. (2001) showed that the problem is strongly NP-hard even if the cost function is uniform, while Bárász et al. (2004) showed that the problem for a measurement "$\lambda^+(S, v) \geq \ell$ and $\lambda^-(S, v) \geq k$" and uniform costs can be solved in polynomial time, where $\lambda^+(S, v)$ (resp., $\lambda^-(S, v)$) is the maximum number of edge-disjoint directed paths from S to v (resp., from v to S).

Ito et al. (2002) treated the source location problem for undirected graphs with unit capacities, a measurement "$\kappa(S, v) \geq k$ and $\lambda(S, v) \geq \ell$ for all $v \in V - S$", and uniform costs $c(v) = 1$, $v \in V$. They presented an $O(m+n^2+n\min\{m,\ell n\}\min\{\ell,n\})$ time algorithm for $k\le 2$ and showed the NP-hardness of the problem for $k\ge 3$ even if $\ell=0$, where $n=|V|$, $m=|\{\{u,v\}\mid (u,v)\in E\}|$.

Thus, the problems with $\psi=\kappa$ are intractable, but Nagamochi et al. (2001) showed that the problem with $\psi=\hat{\kappa}$ and uniform demands $d(v)=k$, $v\in V$ can be solved in polynomial time. For this problem, they gave an $O(\min\{k,\sqrt{n}\}nm)$ time algorithm for digraphs and an $O(\min\{k,\sqrt{n}\}kn^2)$ time algorithm for undirected graphs (notice that if $\psi=\kappa$ or $\psi=\hat{\kappa}$ then edge capacities are assumed to be unit without affecting the problem). Furthermore, they showed that the source location problem for a measurement "$\hat{\kappa}^+(S,v)\ge\ell$ and $\hat{\kappa}^-(S,v)\ge k$" in digraphs can be solved in polynomial time, where $\hat{\kappa}^+(S,v)$ (resp., $\hat{\kappa}^-(S,v)$) is the maximum number of vertex-disjoint directed paths from S to v (resp., from v to S) such that no two paths meet at the same vertex in S. For the problem with $\psi=\hat{\kappa}$, uniform costs c, and general demands d in undirected graphs, Ishii et al. (2003) gave a linear time algorithm in the case of $d\in\{0,1,2,3\}$ and showed that it is NP-hard if there exists a vertex $v\in V$ with $d(v)\ge 4$.

In this paper, we show that the problem with $\psi=\hat{\kappa}$ and general demands $d\in\{0,1,2,3\}$ in undirected graphs is polynomially solvable even if the cost function c is general. By this, we clear the border between NP-hard and polynomially solvable classes of the problem with $\psi=\hat{\kappa}$ in undirected graphs.

We here summarize our method, after reviewing the existing algorithms for the problem with $\psi=\hat{\kappa}$ in undirected graphs. Nagamochi et al. (2001) showed that the problem with uniform demands enjoys a matroidal property and an optimal solution can be found by a greedy method. On the other hand, the problem with general demands does not satisfy such a good property. However, for the problem with uniform costs and $d\in\{0,1,2,3\}$, Ishii et al. (2003) showed that the cardinality of a minimal feasible solution S' obtained by a greedy method is at most twice the optimal for almost all instances. Based on the information on the S', their method finds an optimal solution by replacing some two vertices in S' with one vertex. In this paper, for the problem with general costs and $d\in\{0,1,2,3\}$, our method first finds a minimal feasible solution S' by the same greedy method as one in (Ishii et al. 2003). Based on the information on S', we show that we can reduce the problem to some special case of the hitting set problem (Garey and Johnson 1979), which can be solved by computing the weighted matroid intersection problems (Edmonds 1970) $poly(|V|,|E|)$ times, where $poly(|V|,|E|)$ denotes some polynomial in $|V|$ and $|E|$.

The rest of the paper is organized as follows. Some definitions and preliminaries are described in Section 2. Also in Section 2, we state our main result that the problem with general costs and $d\in\{0,1,2,3\}$ is polynomially solvable. In Section 3, we describe an algorithm for solving the problem, prove its correctness, and discuss the time complexity of our algorithm. Finally, we give some concluding remarks in Section 4.

2 Preliminaries

Let $G=(V,E)$ be a simple undirected graph with a set V of *vertices* and a set E of *edges*, where we denote $|V|$ by n and $|E|$ by m. A singleton set $\{x\}$ may be simply written as x, and "$\subset$" implies proper inclusion while "$\subseteq$" means "$\subset$" or "$=$". A vertex set and an edge set of graph G is denoted by $V(G)$ and $E(G)$, respectively. For a vertex subset $V'\subseteq V$, $G[V']$ means the subgraph induced by V'. For a vertex set $X\subseteq V$, $N_G(X)$ is defined as a set of all vertices in $V-X$ which are adjacent to some of vertices in X. A *partition* $\mathcal{X}=\{X_1,\ldots,X_p\}$ of the vertex set V means a family of nonempty mutually disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset V' of V. For a vertex set $Y\subseteq V$ and a family $\mathcal{X}$ of vertex sets, Y *covers* $\mathcal{X}$ if each $X\in\mathcal{X}$ satisfies $X\cap Y\ne\emptyset$.

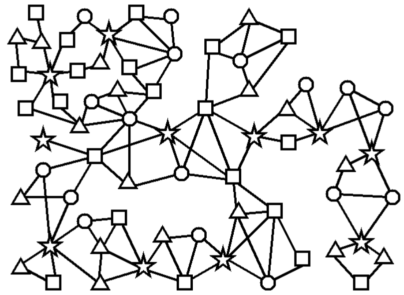
For a vertex $v\in V$ and a vertex set $X\subseteq V-\{v\}$ in G, we denote by $\hat{\kappa}_G(X,v)$ the maximum number of vertex-disjoint paths from v to X such that no two paths meet at the same vertex in X. For a vertex $v\in V$ and a vertex set $X\subset V$ with $v\in X$, let $\hat{\kappa}_G(X,v)=\infty$. By Menger's theorem, the following lemma holds.

Lemma 2.1 *For a vertex $v\in V$ and a vertex set $X\subset V-\{v\}$, $\hat{\kappa}_G(X,v)\ge k$ holds if and only if $|N_G(W)|\ge k$ holds for every vertex set $W\subseteq V-X$ with $v\in W$.* □

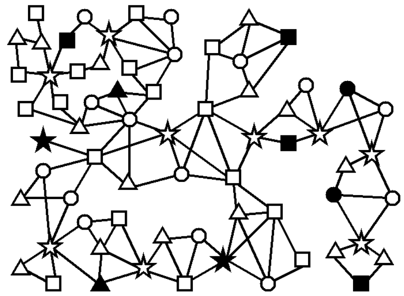
In this paper, each vertex $v\in V$ in $G=(V,E)$ has a *demand* $d(v)$ of nonnegative integer and a *cost* $c(v)$ of nonnegative real. For a cost function $c:V\to R_+$ and a set $X\subseteq V$ of vertices, $c(X)$ is defined as $\sum_{v\in X}c(v)$. A vertex set $S\subseteq V$ is called a *source set* if it satisfies

$$\hat{\kappa}_G(S,v)\ge d(v) \text{ for all vertices } v\in V-S, \quad (1)$$

and we call each vertex $v\in S$ a *source*. In this paper, we consider the following source location problem with local k-vertex-connectivity requirements in an



(a)



(b)

Figure 1: Illustration of an instance of 3LV-CSLP. (a) An initial graph $G=(V,E)$, where each vertex $v\in V$ with $d(v)=0,1,2,3$ is drawn as a square, a triangle, a circle, and a star, respectively, and the cost associated with each vertex is omitted. (b) A set S of black vertices is a source set; there are at least $d(v)$ vertex-disjoint paths between S and each vertex $v\in V-S$ such that no two paths meet at the same vertex in S.

undirected graph (shortly, kLV-CSLP). Fig. 1 gives an instance of 3LV-CSLP.

Problem 2.2 (*kLV-CSLP*)
Input : *An undirected graph $G = (V, E)$, a demand function $d : V \to \{0, 1, \ldots, k\}$, and a cost function $c : V \to R_+$.*
Output : *A source set $S \subseteq V$ minimizing $c(S)$.*

The main result of this paper is described as follows.

Theorem 2.3 *Given an undirected graph $G = (V, E)$, a demand function $d : V \to \{0, 1, 2, 3\}$, and a cost function $c : V \to R_+$, 3LV-CSLP can be solved in $O(n^4(\log n)^2)$ time.* □

In the rest of this section, we introduce several properties for a general kLV-CSLP, which will be used in the subsequent sections. For a vertex set $X \subseteq V$, $d(X)$ denotes the maximum demand among all vertices in X, i.e., $d(X) = \max_{v \in X} d(v)$. A vertex subset $W \subseteq V$ with $d(W) > |N_G(W)|$ is called a *deficient set*. We have the following property by Lemma 2.1.

Lemma 2.4 *A vertex set $S \subseteq V$ satisfies $W \cap S \neq \emptyset$ for every deficient set W if and only if S is a source set.* □

For a vertex $v \in V$, a deficient set $W \subseteq V$ with $v \in W$ is called a *minimal deficient set with respect to $v \in V$*, if no vertex set $W' \subset W$ with $v \in W'$ is a deficient set.

Lemma 2.5 (Ishii et al. 2003) *For a vertex $v \in V$, every minimal deficient set W with respect to $v \in W$ induces a connected graph if $d(v) > |N_G(W)|$ holds.* □

Lemma 2.6 *Each minimal deficient set W with respect to a vertex v satisfies $|N_G(W)| = d(v) - 1$ if $|W| \geq 2$ and $d(v) > |N_G(W)|$ hold.*

Proof. Let W be a deficient set with $|W| \geq 2$, $v \in W$, and $|N_G(W)| < d(v) - 1$. For a vertex $u \in W - \{v\}$, $W' = W - \{u\}$ satisfies $|N_G(W')| \leq |N_G(W)| + 1$ and hence it is also a deficient set with $v \in W'$. □

Lemma 2.7 *Let W be a minimal deficient set with respect to $v \in W$ with $d(v) > |N_G(W)|$. Then for each vertex $x \in N_G(W)$, $(x, v) \in E(G)$ holds or $G[W \cup \{x\}]$ has at least two vertex-disjoint paths between v and x.*

Proof. Otherwise there exists a partition $\{W_1, y, W_2\}$ of $W \cup \{x\}$ such that we have $v \in W_1$, $x \in W_2$, and $N_G(W_1) \cap W = \{y\} = N_G(W_2) \cap W$. By $|N_G(W_1)| \leq |N_G(W)|$, W_1 is also a deficient set, contradicting the minimality of W. □

For two vertex sets X and Y, we say that X and Y *intersect* each other, if none of $X \cap Y$, $X - Y$, and $Y - X$ is empty. For two vertex sets X and Y with $X \cap Y \neq \emptyset$, the following holds.

$$|N_G(X)| + |N_G(Y)| \geq |N_G(X \cap Y)| + |N_G(X \cup Y)|. \tag{2}$$

Lemma 2.8 *Let W_i, $i = 1, 2$ be a minimal deficient set with respect to $w_i \in W_i$ with $d(w_i) > |N_G(W_i)|$ and $|N_G(W_i)| \leq 2$. If W_1 and W_2 intersect each other and $N_G(W_1 \cup W_2) \neq \emptyset$ holds, then one of the following (i)–(iv) hold.*
(i) $|N_G(W_1 \cup W_2)| = 1$, $N_G(W_1) \subseteq W_2$, and $N_G(W_2) - W_1 \neq \emptyset$.
(ii) $|N_G(W_1 \cup W_2)| = 1$, $N_G(W_2) \subseteq W_1$, and $N_G(W_1) - W_2 \neq \emptyset$.
(iii) $|N_G(W_1 \cup W_2)| = 1$ and $N_G(W_1) - W_2 = N_G(W_2) - W_1 \neq \emptyset$.
(iv) $|N_G(W_1 \cup W_2)| = 2$, $|N_G(W_1)| = |N_G(W_2)| = 2$, $N_G(W_1) \cap W_2 = \{w_2\}$, and $N_G(W_2) \cap W_1 = \{w_1\}$.

Proof. Lemma 2.5 says that $W_1 \cap N_G(W_2) \neq \emptyset \neq W_2 \cap N_G(W_1)$ holds. Hence we have $|N_G(W_1 \cap W_2)| \geq 2$. By (2), $|N_G(W_1 \cup W_2)| \leq 2$ holds. The cases of $|N_G(W_1 \cup W_2)| = 1$ imply (i), (ii), or (iii). Assume that $|N_G(W_1 \cup W_2)| = 2$ holds. Then (2) implies that $|N_G(W_1)| = |N_G(W_2)| = |N_G(W_1 \cap W_2)| = 2$ and $|N_G(W_1) \cap W_2| = |N_G(W_2) \cap W_1| = 1$ hold. By Lemma 2.6, we have $d(w_1) = d(w_2) = 3$. From the minimality of W_i, $w_i \in W_i - W_j$ holds for $\{i, j\} = \{1, 2\}$. By Lemma 2.7, we have $\{w_i\} = W_i \cap N_G(W_j)$. □

3 Algorithm

In this section, we give an algorithm for solving 3LV-CSLP. If a given graph is disconnected, then we can consider the problem separately for each connected component. Hence we suppose that G is a connected graph. Also assume that there exists a vertex $v \in V$ with $d(v) \geq 2$ since the problem with $d : V \to \{0, 1\}$ is trivial. Here we propose an algorithm, named 3-LVC_CSLP(x), for finding a source set S such that S contains a given vertex $x \in V$ and $c(S)$ is minimized. Note that an optimal solution to 3LV-CSLP can be obtained by executing algorithm 3-LVC_CSLP(x) for each vertex $x \in V$.

We first sketch algorithm 3-LVC_CSLP(x), which consists of two steps. The first step is a greedy method to find a minimal feasible solution S_0 and a family $\mathcal{W}_0$ of minimal deficient sets with respect to some $s \in S_0$. We start with a source set $S_0 = V$ and a family $\mathcal{W}_0 := \emptyset$ of minimal deficient sets, and pick up vertices $v \in V - \{x\}$, one by one, in increasing order of their demands. If $S_0 - \{v\}$ remains to be a source set, update $S_0 := S_0 - \{v\}$, and otherwise we have a minimal deficient set W with respect to v with $W \cap S_0 = \{v\}$ and update $\mathcal{W}_0 := \mathcal{W}_0 \cup \{W\}$ (note that Lemma 2.4 says that such W exists).

In the second step, we reduce the problem to a problem of finding a vertex set covering specified deficient sets obtained from $\mathcal{W}_0$. First, we decompose $V - \{x\}$ into a subpartition $\{X_1, \ldots, X_p\}$ of $V - \{x\}$ based on the information on S_0 and $\mathcal{W}_0$. For each X_i, we pick up $O(|X_i|)$ pairs $\{\mathcal{Y}_i^{j1}, \mathcal{Y}_i^{j2}\}$ of subpartitions of X_i which consist of specified deficient sets obtained from $\mathcal{W}_0$, compute a vertex set S_i^j with the minimum cost covering $\mathcal{Y}_i^{j1}$ and $\mathcal{Y}_i^{j2}$, and obtain the vertex set S_i with $c(S_i) = \min_j\{c(S_i^j)\}$. Note that the problem of finding a vertex set with the minimum cost covering two subpartitions is a weighted matroid intersection problem (Edmonds 1970), and it can be solved in $O(|X_i|^2(log|X_i|)^2)$ time (Nagamochi & Ibaraki 2000). Finally, we output $S_1 \cup \cdots \cup S_p \cup \{x\}$ as an optimal solution. The key point of this method is that we can obtain an optimal source set in the original problem, by combining the vertex x and vertex sets S_i obtained from V_i locally.

A more precise description of Step I in the algorithm is given as follows. Step II is very complicated, and hence the details will be mentioned after describing Step I and analyzing properties of S_0 and $\mathcal{W}_0$.

Algorithm 3-LVC_CSLP(x)
Input: An undirected connected graph $G = (V, E)$, a demand function $d : V \to \{0, 1, 2, 3\}$, a cost function $c : V \to R_+$, and a vertex $x \in V$.
Output: A source set S with $x \in S$ minimizing $c(S)$.
Step I (I-0) Number vertices of V such as $d(v_1) \leq \cdots \leq d(v_n)$.
(I-1) Initialize $j := 1$, $S_0 := V$, and $\mathcal{W}_0 := \emptyset$.
(I-2) If $v_j = x$ holds, then go to Step (I-4).

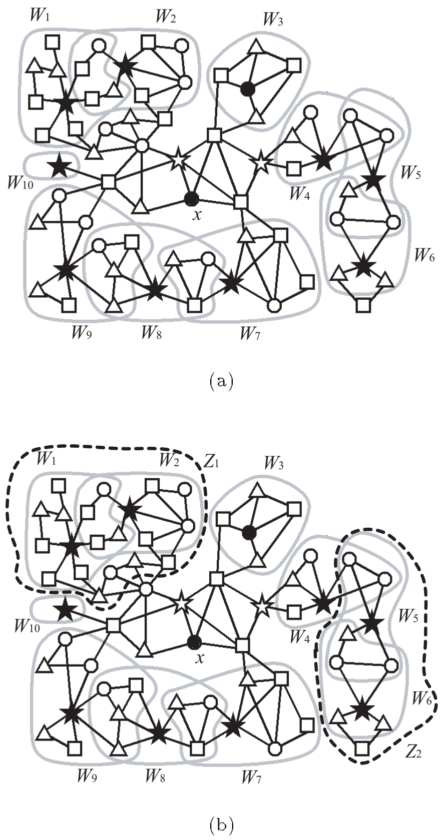


Figure 2: (a) Illustration of a source set S_0 and a family $\mathcal{W}_0 = \{W_1, W_2, \ldots, W_{10}\}$ of minimal deficient sets, which are obtained by applying Step I of algorithm 3-LVC_CSLP(x) to G in Fig. 1(a). Each vertex in S_0 is drawn as black vertices. (b) Illustration of a vertex set Z_1 for the chain $\mathcal{X}_1 = \{W_1, W_2\}$ and a vertex set Z_2 for the chain $\mathcal{X}_2 = \{W_4, W_5, W_6\}$.

(I-3) If $S_0 - \{v_j\}$ satisfies (1) then let $S_0 := S_0 - \{v_j\}$. Otherwise select a minimal deficient set $W' \subseteq V - (S - \{v_j\})$ with respect to v_j, and let $\mathcal{W}_0 := \mathcal{W}_0 \cup \{W'\}$.
(I-4) If $j < n$, then $j := j + 1$ and go to Step (I-2). Otherwise go to Step II.
Step II Find a vertex set Y^* with the minimum cost which covers a family $\mathcal{Y}^*$ of specified deficient sets obtained from $\mathcal{W}_0$, and output $Y^* \cup \{x\}$ as an optimal solution. The details are given in Section 3.2. □

Let S_0 and $\mathcal{W}_0$ be a source set and a family of the corresponding deficient sets obtained after v_n is checked in Step I, respectively. Fig. 2(a) shows S_0 and $\mathcal{W}_0$ obtained by applying Step I to G in Fig. 1(a). Note that $x \in S_0$ holds. Then S_0 and $\mathcal{W}_0$ can be characterized as follows.

Definition 3.1 *For a source set S, we say that a deficient set W has* property (P) *with respect to S, if W satisfies $W \cap S = \{s\}$ and $d(W) = d(s)$, and it is minimal with respect to s. Moreover, we say that a family $\mathcal{W}$ of deficient sets has* property (P) with respect to S *if each $W \in \mathcal{W}$ has property (P) with respect to S and every two sets W_1 and W_2 in $\mathcal{W}$ satisfy $W_1 \cap W_2 \cap S = \emptyset$.* □

Lemma 3.2 *Let S_0 and $\mathcal{W}_0$ be a source set and a family of minimal deficient sets obtained by Step I of algorithm 3-LVC_CSLP(x), respectively. Then the following statements (i)–(iii) hold. (i) $d(s) \in \{2, 3\}$ holds for each $s \in S_0 - \{x\}$. (ii) We have $\{x\} = S_0 - \bigcup_{W \in \mathcal{W}_0} W$. (iii) $\mathcal{W}_0$ has property (P) with respect to S_0.*

Proof. (i) By $d(V) \geq 2$, each vertex v with $d(v) \leq 1$ is always deleted from the current S_0 at Step I-3. (ii) By Steps I-2 and I-3, we can see that no $W \in \mathcal{W}_0$ contains x, and each $s \in S_0 - \{x\}$ is contained in some deficient set in $\mathcal{W}_0$. (iii) It suffices to show that each $W \in \mathcal{W}_0$ has property (P) with respect to S_0, since Step I-3 implies that each $s \in S_0 - \{x\}$ is contained in exactly one set in $\mathcal{W}_0$. At Step I-3, assume that v_j cannot be deleted. Then the current solution S_0 satisfies $W' \cap S_0 = \{v_j\}$. Then all vertices in $W' - \{v_j\}$ have been already deleted, and $d(v_j) = \max\{d(v) \mid v \in W'\}$ holds by the sorting in Step I-0. Hence the set $W' \in \mathcal{W}_0$ and the vertex $v_j \in S_0$ satisfy $W' \cap S_0 = \{v_j\}$ and $d(W') = d(v_j)$. □

After analyzing properties of a source set S and a family $\mathcal{W}$ of deficient sets which has property (P) with respect to S in Section 3.1, we give the procedure of Step II for the details in Section 3.2.

3.1 Property (P)

Through this section, let S be a source set and a family $\mathcal{W}$ of minimal deficient sets have property (P) with respect to S. Let $S_1 = S - \bigcup_{W \in \mathcal{W}} W$ ($S_1 = \emptyset$ may hold). Assume that each $s \in S - S_1$ satisfies $d(s) \in \{2, 3\}$. Note that S_0 and $\mathcal{W}_0$ obtained in Step I of algorithm 3-LVC_CSLP(x) correspond to S and $\mathcal{W}$ with $S_1 = \{x\}$, respectively. We here show several lemmas, some of which generalize observations given in (Ishii et al. 2003) slightly.

First, we observe the properties of deficient sets in $\mathcal{W}$ which intersect each other. The following lemma shows that each vertex is contained in at most two sets in $\mathcal{W}$ and each set in $\mathcal{W}$ intersects at most two sets in $\mathcal{W}$, in the case of $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$.

Lemma 3.3 *Let S be a source set and a family $\mathcal{W}$ of minimal deficient sets W_i have property (P) with respect to S such that $S \cap W_i = \{s_i\}$.*
(i) If $|W_j \cap N_G(W_i)| = 1$ holds for $W_i, W_j \in \mathcal{W}$, then $W_j \cap N_G(W_i) = \{s_j\}$ holds.
(ii) Assume that $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$ hold. Let $W_i \in \mathcal{W}$. If there exist two distinct $W_h, W_j \in \mathcal{W} - \{W_i\}$ with $W_i \cap W_h \neq \emptyset \neq W_i \cap W_j$, then we have $W_h \cap W_i \cap W_j = \emptyset$ and $N_G(W_i) = \{s_h, s_j\}$ (hence, the number of $W \in \mathcal{W} - \{W_i\}$ with $W_i \cap W \neq \emptyset$ is at most two).

Proof. (i) By the property (P), we have $s_j \notin W_i$. By Lemma 2.5, $N_G(W_j) \cap W_i \neq \emptyset$ holds. Lemma 2.7 and these imply that $W_j \cap N_G(W_i) = \{s_j\}$ holds.
(ii) For each $W_i \in \mathcal{W}$, we have $|N_G(W_i)| \in \{1, 2\}$ by Lemmas 2.6 and 3.2(i). Denote $N_G(W_i)$ and $N_G(W_j)$ by $\{x_i, y_i\}$ and $\{x_j, y_j\}$, respectively (possibly $x_i = y_i$ or $x_j = y_j$ hold). We first claim that $N_G(W_i \cup W_j \cup W_h) \neq \emptyset$ holds. If $N_G(W_i \cup W_j \cup W_h) = \emptyset$ would hold, i.e., $V = W_i \cup W_j \cup W_h$ would hold, then we would have $|S| = 3$ by the property (P), which contradicts the assumption that $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$ hold. Assume that there exist two distinct $W_h, W_j \in \mathcal{W} - \{W_i\}$ with $W_i \cap W_h \neq \emptyset \neq W_i \cap W_j$. By Lemma 2.5, we have $W_j \cap N_G(W_i) \neq \emptyset \neq W_h \cap N_G(W_i)$.

Assume that $W_j \cap \{x_i, y_i\} = \{x_i\}$ holds without loss of generality. Then we have $x_i = s_j$ by (i). Since $s_j \notin W_h$ holds by the property (P), we have $W_h \cap \{x_i, y_i\} = \{y_i\}$, from which we have $y_i = s_h$. If another $W_k \in \mathcal{W} - \{W_i, W_j, W_h\}$ satisfies $W_k \cap W_i \neq \emptyset$, then the property (P) implies $s_k \in W_k - W_i$ and $N_G(W_i) \cap W_k = \{s_h, s_j\} \cap W_k = \emptyset$, from which $G[W_k]$ is not connected, contradicting Lemma 2.5. So there is no set $W \in \mathcal{W} - \{W_i, W_j, W_h\}$ with $W \cap W_i \neq \emptyset$. From $N_G(W_i) \subseteq W_h \cup W_j$, let $y_j \in N_G(W_i \cup W_j \cup W_k)$ and $x_j \in W_i$ without loss of generality. Again by (i) and $N_G(W_j) \cap W_i = \{x_j\}$, we see $x_j = s_i$. Hence by the property (P), we have $s_i = x_j \notin W_h$. Therefore, $\{s_j, s_i\} = N_G(W_i \cap W_j) \subseteq V - W_h$ and Lemma 2.5 imply that $W_i \cap W_j \cap W_h = \emptyset$ holds.

Assume that $\{x_i, y_i\} \subseteq W_j$ and $\{x_i, y_i\} \subseteq W_h$ hold. We have $W_j \cap W_h \neq \emptyset$ and $N_G(W_j) \cap W_i \neq \emptyset \neq N_G(W_h) \cap W_i$ by Lemma 2.5. Without loss of generality, we can assume $y_j \in N_G(W_i \cup W_j \cup W_h)$ by $N_G(W_i) \subseteq W_j \cup W_h$. By $N_G(W_j) \cap W_i \neq \emptyset$, we have $x_j \in N_G(W_j) \cap W_i$. Then we have $s_i = x_j$ by (i). Hence $W_h \cap N_G(W_j) = \emptyset$ holds, since the property (P) implies that $s_i = x_j \notin W_h$. This contradicts Lemma 2.5. □

We decompose $\mathcal{W}$ into subfamilies $\mathcal{X}_i$ in the following manner. Let $S'_i \subseteq S - S_1$, $i = 1, \ldots, q$ be a connected component of the graph $H = (S - S_1, \{(s_j, s_k) \mid \{W_j, W_k\} \subseteq \mathcal{W},\ W_j \cap W_k \neq \emptyset,\ S \cap W_j = \{s_j\},\ S \cap W_k = \{s_k\}\})$. A family of deficient sets in $\mathcal{W}$ corresponding to the sources in S'_i is denoted by $\mathcal{X}_i$. In Fig. 2, each of $\{W_1, W_2\}$, $\{W_3\}$, $\{W_4, W_5, W_6\}$, $\{W_7, W_8, W_9\}$, and $\{W_{10}\}$ corresponds to $\mathcal{X}_i$ for some i.

We here define a family of deficient sets called a *chain* as follows.

Definition 3.4 *A family $\mathcal{W}' = \{W_1, \ldots, W_t\}$ $(t \geq 1)$ of deficient sets is called a chain if it satisfies the following conditions (a) and (b).*
(a) $W_i \cap W_{i+1} \neq \emptyset$ holds for $i = 1, \ldots, t-1$ if $t \geq 2$.
(b) $W_i \cap W_h = \emptyset$ holds for two distinct $i, h \in \{1, 2, \ldots, t\}$ with $2 \leq |i - h| \leq t - 2$ if $t \geq 2$. □

In Fig. 2, each $\mathcal{X}_i$ is a chain. Lemma 3.3 indicates that each $\mathcal{X}_i$ is a chain in the case of $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$.

Lemma 3.5 *Let S be a source set and a family $\mathcal{W}$ of minimal deficient sets have property (P) with respect to S in G. If $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$ hold, then each $\mathcal{X}_i$ is a chain.* □

Let $\mathcal{X}_i = \{W^i_1, W^i_2, \ldots, W^i_{|\mathcal{X}_i|}\}$ and $W^i_j \cap S = \{s^i_j\}$, where $W^i_{j-1} \cap W^i_j \neq \emptyset$ holds for each $j \in \{2, 3, \ldots, |\mathcal{X}_i|\}$. Lemma 3.3 implies that $N_G(W^i_j) = \{s^i_{j-1}, s^i_{j+1}\}$, $j \in \{2, 3, \ldots, |\mathcal{X}_i| - 1\}$, holds. Chains can be divided into three types as follows.

Definition 3.6 *Let S be a source set and a family $\mathcal{W}$ of minimal deficient sets have property (P) with respect to S in G. Let $\mathcal{X} = \{W_1, \ldots, W_t\} \subseteq \mathcal{W}$ be a chain satisfying (a) and (b) in Definition 3.4 and $W_i \cap S = \{s_i\}$ for each i. Then $\mathcal{X}$ is called* type (A) *if it satisfies the following conditions (i) and (ii),* type (B) *if it satisfies neither (i) nor (ii), and* type (C) *otherwise. (In the case of type (C), assume that $\mathcal{X}$ satisfies (i) and does not satisfy (ii) without loss of generality.)*
(i) $t \geq 2$. There exists $Z'_1 \subseteq V$ with $W_1 \cup W_2 \subseteq Z'_1$, $|N_G(Z'_1)| = 1$, $N_G(Z'_1) \cap N_G(W_2) \neq \emptyset$ and $Z'_1 \cap S = \{s_1, s_2\}$.
(ii) $t \geq 2$. There exists $Z'_t \subseteq V$ with $W_{t-1} \cup W_t \subseteq Z'_t$, $|N_G(Z'_t)| = 1$, $N_G(Z'_t) \cap N_G(W_{t-1}) \neq \emptyset$ and $Z'_t \cap S = \{s_{t-1}, s_t\}$ (note that if $t \geq 3$ holds then we have $N_G(Z'_1) = \{s_3\}$ and $N_G(Z'_t) = \{s_{t-2}\}$ by $N_G(W_2) = \{s_1, s_3\}$, $N_G(W_{t-1}) = \{s_{t-2}, s_t\}$, and Lemma 3.3). □

In Fig. 2(b), $\{W_4, W_5, W_6\}$ is a chain of type (C) with $Z'_1 = Z_2$ and each of $\{W_1, W_2\}$, $\{W_3\}$, $\{W_7, W_8, W_9\}$, and $\{W_{10}\}$ is of type (B). Note that if $\mathcal{X}_i$ is a chain of type (A) or a chain of type (B) with $W^i_1 \cap W^i_{|\mathcal{X}_i|} \neq \emptyset$, then Lemma 3.3(ii) implies that we have $\bigcup_{W \in \mathcal{X}_i} W \supseteq S$, i.e., $\mathcal{W} = \mathcal{X}_i$. We here assume that each chain is a chain of type (B) with $W^i_1 \cap W^i_{|\mathcal{X}_i|} = \emptyset$ or type (C) (the above two special cases can be treated similarly, however we omit the details). Note that the case of $S_1 \neq \emptyset$ always satisfies this assumption. For each chain $\mathcal{X}_i$ of type (C), let Z_i be a vertex set corresponding to Z'_1 in Definition 3.6 (i); $W^i_1 \cup W^i_2 \subseteq Z_i$, $|N_G(Z_i)| = 1$, $N_G(Z_i) \cap N_G(W^i_2) \neq \emptyset$ and $Z_i \cap S = \{s^i_1, s^i_2\}$. For each chain $\mathcal{X}_i$ of type (B) satisfying $|\mathcal{X}_i| \leq 2$, $|W^i_1| \geq 2$, and $d(\bigcup_{W \in \mathcal{X}_i} W) = 3$, let Z_i be a vertex set with $\bigcup_{W \in \mathcal{X}_i} W \subseteq Z_i$, $|N_G(Z_i)| = 1$, and $S \cap Z_i = S \cap (\bigcup_{W \in \mathcal{X}_i} W)$ such that no vertex set $Z' \subset Z_i$ satisfies this property if exists, $Z_i = \emptyset$ otherwise (note that Z_i is uniquely determined). For any other chain $\mathcal{X}_i$ of type (B), let $Z_i = \emptyset$. Fig. 2(b) shows Z_1 and Z_2 for the chains $\mathcal{X}_1 = \{W_1, W_2\}$ and $\mathcal{X}_2 = \{W_4, W_5, W_6\}$.

In the sequel, we show that we can construct a source set $\bigcup_i S_i$ by combining a set S_i of vertices covering some family of deficient sets constructed from $\mathcal{X}_i$. However, $\bigcup_i S'_i$ obtained from choosing S'_i directly as a vertex set covering $\mathcal{X}_i$ may not be a source set. For example, in Fig. 2(b), a vertex $v \in W_1 \cap W_2$ can cover the chain $\{W_1, W_2\}$, but $Z_1 - v$ still remains a deficient set. To overcome this, we define deficient sets not in $\mathcal{W}$ to be covered for each chain as follows. For a chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ and $d(s^i_1) = 2$ (resp., $d(s^i_1) = 3$) and a vertex $u \in W^i_1 - \{s^i_1\}$, let $Z_i(u) \subseteq Z_i - \{u\}$ denote a minimal deficient set with respect to s^i_2 (resp., s^i_1) and $Z_i(s^i_1) = \emptyset$ (note that each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ satisfies $d(\bigcup_{W \in \mathcal{X}_i} W) = 3$). Then we can observe that $Z_i(u)$ is uniquely determined.

Lemma 3.7 *Let S be a source set and a family $\mathcal{W}$ of minimal deficient sets have property (P) in G. For a chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ and a vertex $u \in W^i_1 - \{s^i_1\}$, $Z_i(u)$ is unique.*

Proof. See Appendix. □

For each chain $\mathcal{X}_i$ with $Z_i = \emptyset$, let $\mathcal{Y}^+_i = \{W^i_{2j-1} \mid j = 1, 2, \ldots, \lceil |\mathcal{X}_i|/2 \rceil\}$ and $\mathcal{Y}^-_i = \{W^i_{2j} \mid j = 1, 2, \ldots, \lfloor |\mathcal{X}_i|/2 \rfloor\}$. For each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ and a vertex $u \in W^i_1$, we define two families $\mathcal{Y}^+_i(u)$ and $\mathcal{Y}^-_i(u)$ of deficient sets as the following (a) and (b).

(a) $\mathcal{Y}^+_i(u) = \{W^i_{2j-1} \mid j = 2, \ldots, \lceil |\mathcal{X}_i|/2 \rceil\}$ and $\mathcal{Y}^-_i(u) = \{W^i_{2j} \mid j = 2, \ldots, \lfloor |\mathcal{X}_i|/2 \rfloor\} \cup \{Z_i(u)\}$ if $d(s^i_1) = 2$.

(b) $\mathcal{Y}^+_i(u) = \{W^i_{2j-1} \mid j = 2, \ldots, \lceil |\mathcal{X}_i|/2 \rceil\} \cup \{Z_i(u)\}$ and $\mathcal{Y}^-_i(u) = \{W^i_{2j} \mid j = 1, 2, \ldots, \lfloor |\mathcal{X}_i|/2 \rfloor\}$ if $d(s^i_1) = 3$.

The following lemma shows that given a vertex $u_i \in W^i_1$ for each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$, we can find a source set by finding a vertex set Y which covers $\bigcup_{i: Z_i \neq \emptyset} (\mathcal{Y}^+_i(u_i) \cup \mathcal{Y}^-_i(u_i)) \cup \bigcup_{i: Z_i = \emptyset} (\mathcal{Y}^+_i \cup \mathcal{Y}^-_i)$.

Lemma 3.8 *Let S be a source set and a family $\mathcal{W}$ of minimal deficient sets have property (P) with respect to S in G. Assume that $|S| \geq 5$ or $S_1 \neq \emptyset$ hold (note that $S_1 \neq \emptyset$ implies that $V - \bigcup_{i: \mathcal{X}_i \subseteq \mathcal{W}} X_i \supseteq$*

$S_1 \neq \emptyset$ *hold, where* $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup \{Z_i\}$ *for a chain* $\mathcal{X}_i$*). Let* u_i *be a vertex chosen arbitrarily from* W_1^i *in each chain* $\mathcal{X}_i \subseteq \mathcal{W}$ *with* $Z_i \neq \emptyset$*. Let* Y *be a vertex set which covers any deficient set in* $\bigcup_i \mathcal{Y}_i'$*, where* $\mathcal{Y}_i' = \mathcal{Y}_i^+(u_i) \cup \mathcal{Y}_i^-(u_i)$ *if* $Z_i \neq \emptyset$*, and* $\mathcal{Y}_i' = \mathcal{Y}_i^+ \cup \mathcal{Y}_i^-$ *otherwise. Then* $S^* = S_1 \cup Y \cup \{u_i \mid Z_i \neq \emptyset\}$ *is a source set.*

Proof. See Appendix. □

In addition to the assumption of Lemma 3.8, let us assume that $V \neq \bigcup_{i:\mathcal{X}_i \subseteq \mathcal{W}} X_i$ if $|S| \leq 4$, where $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup \{Z_i\}$. Based on Lemma 3.8, we next show that we can find a source set with the minimum cost among all source sets containing S_1, by finding a vertex set with the minimum cost covering a family of sets to be covered for each chain (the proof is given as the proof of Lemma 3.11). For this, we here assume that for any chain $\mathcal{X}_i$ with $|\mathcal{X}_i| = 1$, $d(W_1^i) = 3$, and $|W_1^i| \geq 2$, if there exists $W \in \mathcal{W} - \{W_1^i\}$ with $N_G(W_1^i) \subseteq W$, then any set X with $|N_G(X)| \leq 1$ and $W_1^i \subseteq X$ satisfies $W \cap S \subseteq X$. The following Lemma 3.9 shows that this assumption is possible; if $\mathcal{W}$ violates the assumption, then we can obtain another family $\mathcal{W}'$ satisfying the assumption by replacing some sets in $\mathcal{W}$. Under this assumption, Lemma 3.10 shows that the family of vertex sets $(\bigcup_{W \in \mathcal{X}_i} W) \cup \{Z_i\}$ is a subpartition of V.

Lemma 3.9 *Let* S *be a source set and a family* $\mathcal{W}$ *of minimal deficient sets have property* (P) *with respect to* S *in* G*. Let* $W_i \in \mathcal{W}$ *satisfy* $d(W_i) = 3$ *and* $|W_i| \geq 2$*. Assume that* $|S| \geq 3$ *or* $V \neq \bigcup_{W \in \mathcal{W}} W$ *hold, that there exists* $W \in \mathcal{W} - \{W_i\}$ *with* $W_i \cap W = \emptyset$ *and* $N_G(W_i) \subseteq W$*, and that there exists a set* X *with* $|N_G(X)| \leq 1$*,* $W_i \subseteq X$*, and* $W \cap S \cap X = \emptyset$*. Then there exists a set* $W' \subseteq W \cup W_i$ *with* $W' \cap W \neq \emptyset$ *and* $S \cap W_i = S \cap W'$ *such that* W' *has property* (P) *with respect to* S*;* $\mathcal{W}' = (\mathcal{W} - \{W_i\}) \cup \{W'\}$ *also has property* (P) *with respect to* S*. Moreover, such a* W' *can be computed in* $O(|W_i \cup W|)$ *time.*

Proof. See Appendix. □

Lemma 3.10 *Let* S *be a source set and a family* $\mathcal{W}$ *of minimal deficient sets have property* (P) *in* G*. For a chain* $\mathcal{X}_i$*, let* $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup \{Z_i\}$*. Assume that* $|S| \geq 5$ *or* $V \neq \bigcup_{i:\mathcal{X}_i \subseteq \mathcal{W}} X_i$ *hold. Then the family of vertex sets* X_i *is a subpartition of* V*.*

Proof. From the construction of $\mathcal{W}_i$, a family of vertex sets $X_i' = (\bigcup_{W \in \mathcal{X}_i} W)$ is a subpartition of V. Note that every two distinct sets Z_i, Z_j are pairwise disjoint, since if $Z_i \cap Z_j \neq \emptyset$ holds, then $|N_G(Z_i)| = |N_G(Z_j)| = 1$ implies that $V = Z_i \cup Z_j$ holds, contradicting that $|S| \geq 5$ or $V \neq \bigcup_{i:\mathcal{X}_i \subseteq \mathcal{W}} X_i$ hold (note that each of Z_i and Z_j contains at most two sources in S). Hence, it suffices to show that each Z_i with $N_G(Z_i) = N_G(X_i)$ and $N_G(Z_i) - N_G(X_i') \neq \emptyset$ is disjoint with any $W \in \mathcal{W} - \mathcal{X}_i$. Let Z_i satisfy $N_G(Z_i) = N_G(X_i)$ and $N_G(Z_i) - N_G(X_i') \neq \emptyset$, and $\{z_i\} = N_G(Z_i)$ (note that $|N_G(Z_i)| = 1$ holds). By $z_i \notin N_G(X_i')$ and the definition of Z_i, we see that $|\mathcal{X}_i| \leq 2$ and $|N_G(X_i')| = 2$ hold and $\mathcal{X}_i$ is of type (B). Let $\{x_1, x_2\} = N_G(X_i')$. Assume by contradiction that some $W \in \mathcal{W} - \mathcal{X}_i$ satisfies $W \cap Z_i \neq \emptyset$. We have $V \neq W \cup Z_i$ since $|S| \geq 5$ or $V \neq \bigcup_{i:\mathcal{X}_i \subseteq \mathcal{W}} X_i$ hold. Hence, $N_G(W) - Z_i \neq \emptyset$ and $|N_G(W) \cap Z_i| = 1$ hold. Let $\{x\} = N_G(W) \cap Z_i$. Now, from the minimality of Z_i and $z_i \notin N_G(X_i')$, we see that there are two vertex-disjoint paths P_1 and P_2 such that P_k, $k = 1, 2$ connects x_k and z_i in $G[Z_i \cup \{z_i\}]$. Hence $x \in X_i'$ and $\{x_1, x_2\} \subseteq W$ hold. Let $x \in W_1^i$ (this does not lose generality because $\mathcal{X}_i$ is of type (B)). Lemma 3.3 implies that $x = s_1^i$ holds. Hence, $x \notin W_2^i$ and $N_G(W) \cap W_2^i = \emptyset$ hold if $|\mathcal{X}_i| = 2$ holds. This implies that if $|\mathcal{X}_i| = 2$ holds, then we have $\{x_1, x_2\} \subseteq N_G(W_1^i) - N_G(W_2^i)$, a contradiction to $N_G(W_1^i) \cap W_2^i \neq \emptyset$. Hence $|\mathcal{X}_i| = 1$ holds. Note that $|N_G(W_1^i)| = |N_G(X_i')| = 2$ and $d(W_1^i) = 3$ hold. Also note that $|W_1^i| \geq 2$ holds from the definition of Z_i. From the construction of Z_i, $S \cap W \cap Z_i = \emptyset$ holds. These contradict the assumption of $\mathcal{W}$ mentioned immediately before Lemma 3.9. □

Lemma 3.11 *Let* S *and* $\mathcal{W}$ *satisfy the assumption of Lemma 3.8. For a chain* $\mathcal{X}_i$ *with* $Z_i \neq \emptyset$ *and a vertex* $u \in W_1^i$*, let* $S_i(u)$ *be a vertex set with the minimum cost which covers* $\mathcal{Y}_i^+(u) \cup \mathcal{Y}_i^-(u)$*, and* S_i^* *be a vertex set* $S_i(u^*) \cup \{u^*\}$ *with* $c(S_i(u^*) \cup \{u^*\}) = \min_{u \in W_1^i} c(S_i(u) \cup \{u\})$*. For a chain* $\mathcal{X}_i$ *with* $Z_i = \emptyset$*, let* S_i^* *be a vertex set with the minimum cost which covers* $\mathcal{Y}_i^+ \cup \mathcal{Y}_i^-$*. Then* $S_1 \cup (\bigcup_i S_i^*)$ *is a source set with the minimum cost among source sets containing* S_1*.*

Proof. Let S_{opt} be a source set with the minimum cost among source sets containing S_1. By Lemma 3.8, $S_1 \cup (\bigcup_i S_i^*)$ is feasible. By Lemma 3.10, it suffices to show that for each chain $\mathcal{X}_i$, we have $c(S_i^*) \leq c(S_{opt} \cap X_i)$, where $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup Z_i$.

Let $\mathcal{X}_i$ be a chain with $Z_i = \emptyset$. From the feasibility of S_{opt}, we have $S_{opt} \cap W \neq \emptyset$ for each $W \in \mathcal{X}_i$. From the minimality of $c(S_i^*)$, we have $c(S_i^*) \leq c(S_{opt} \cap X_i)$.

Let $\mathcal{X}_i$ be a chain with $Z_i \neq \emptyset$. From the feasibility of S_{opt}, we have $S_{opt} \cap W_1^i \neq \emptyset$; let $u' \in S_{opt} \cap W_1^i$. Then by Lemma 3.7, we have a unique deficient set $Z_i(u')$ and a family $\mathcal{Y}_i^+(u') \cup \mathcal{Y}_i^-(u')$ of deficient sets to be covered by any feasible solution. Hence from the minimality of $c(S_i(u'))$, we see that $c(S_i(u')) \leq c((S_{opt} - \{u'\}) \cap X_i)$. □

Before closing this section, we give the following lemma, which will be used to analyze the complexity of the algorithm 3-LVC_CSLP(x).

Lemma 3.12 *Let* S *be a source set and a family* $\mathcal{W}$ *of minimal deficient sets have property* (P) *in* G*. Assume that* $|S| \geq 4$ *or* $V \neq \bigcup_{W \in \mathcal{W}} W$ *hold. Then for a chain* $\mathcal{X}_i$ *with* $Z_i \neq \emptyset$*, each of* $\mathcal{Y}_i^+(u)$ *and* $\mathcal{Y}_i^-(u)$ *is a subpartition of* V *for any* $u \in W_1^i$*.*

Proof. See Appendix. □

3.2 Step II

The procedure of Step II is given as follows. Let $\mathcal{X}_i$, Z_i, W_1^i, $\mathcal{Y}_i^+$, $\mathcal{Y}_i^-$, $\mathcal{Y}_i^+(u)$, and $\mathcal{Y}_i^-(u)$ be defined as Section 3.1, regarding the source set S_0, the family $\mathcal{W}_0$, and the set $\{x\} \subseteq S_0$ as S, $\mathcal{W}$, and S_1, respectively.

Step II (II-0) Execute the following procedure (II-1) and (II-2) for each chain $\mathcal{X}_i \subseteq \mathcal{W}_0$.
(II-1) If $Z_i = \emptyset$ holds, then compute a vertex set S_i^* with the minimum cost which covers $\mathcal{Y}_i^+ \cup \mathcal{Y}_i^-$.
(II-2) Otherwise execute the following procedure (II-2-0) for each $u \in W_1^i$.

(II-2-0) Compute a set $S_i(u)$ with the minimum cost covering $\mathcal{Y}_i^+(u) \cup \mathcal{Y}_i^-(u)$.
(II-2-1) Let S_i^* be a vertex set $S_i(u) \cup \{u\}$ with $c(S_i(u) \cup \{u\}) = \min_{u' \in W_1^i} c(S_i(u') \cup \{u'\})$.
(II-3) Output $\{x\} \cup \bigcup_i S_i^*$ as an optimal solution and halt. □

Through the procedure, $S_1 = \{x_1\} \neq \emptyset$ holds. Lemmas 3.2 and 3.11 and this prove the correctness of algorithm 3-LVC_CSLP(x).

3.3 Complexity

We here analyze the complexity of algorithm 3-LVC_CSLP(x). As shown in (Ishii et al. 2003), Step I can be computed in linear time. We consider the time complexity of Step II. By Lemma 3.12, we can see that for each chain $\mathcal{X}_i$, we compute a vertex set with the minimum cost which covers two subpartitions of X_i at most $|W_1^i|$ times. This problem of covering two subpartitions can be formulated as follows.

Problem 3.13 Input: *A finite set V, a cost function $c : V \to R_+$, and two subpartitions $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of V.*
Output: *A subset S of V minimizing $c(S)$ such that each $Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ satisfies $S \cap Y \neq \emptyset$.* □

In (Nagamochi & Ibaraki 2000, Theorem 8), it was shown that Problem 3.13 can be solved in $O(|V|^2(log|V|)^2)$ time by the minimum cost flow algorithm. Hence, Step II can be implemented to run in $O(\sum_i |W_1^i||X_i|^2(\log |X_i|)^2) = O(n^3(\log n)^2)$ time (note that each chain $\mathcal{X}_i$ obtained in Step I is of type (B) or type (C) by $S_1 = \{x\} \neq \emptyset$). Consequently, the total time complexity of algorithm 3-LVC_CSLP(x) is $O(n^3(\log n)^2)$.

An optimal solution to Problem 2.2 can be obtained by executing algorithm 3-LVC_CSLP(x) for each vertex $x \in V$. Therefore it can be found in $O(n^4(\log n)^2)$ time. Summarizing the argument given so far, Theorem 2.3 is now established.

Remark 1. We can consider a variant of algorithm 3-LVC_CSLP(x), in which a vertex x is not given. As observed in Section 3.1, if a source set S_0 obtained in Step I satisfies $|S_0| \geq 5$, then such algorithm works. However, in the case of $|S_0| \leq 4$, it is difficult to characterize S_0 and $\mathcal{W}_0$ as Section 3.1. □

4 Concluding Remarks

In this paper, given an undirected graph $G = (V, E)$, a demand function $d : V \to \{0, 1, 2, 3\}$, and a cost function $c : V \to R_+$, we have considered the problem of finding a source set $S \subseteq V$ minimizing $c(S)$ for which there exist $d(v)$ mutually vertex-disjoint paths between every vertex $v \in V - S$ and S such that no two paths meet at the same vertex in S. We have shown that the problem can be solved in $O(n^4(\log n)^2)$ time.

For general demands $d \geq 4$, the problem is NP-hard, even if every vertex has a uniform cost, as shown in (Ishii et al. 2003). It is a future work to design approximation algorithms for these problems.

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Appendix

Proof of Lemma 3.7. We first consider the case of $d(s_1^i) = 3$. Assume by contradiction that there exist two distinct minimal deficient sets W' and W'' in $Z_i - \{u\}$ with respect to s_1^i. From $s_1^i \in W' \cap W''$ and the minimality of W' and W'', W' and W'' intersect each other. By $u \notin W' \cup W''$ and the connectedness of G, $N_G(W' \cup W'') \neq \emptyset$ holds. Lemma 2.8 implies that we have $|N_G(W' \cup W'')| = 1$. The connectedness of $G[W_1^i]$, $u \notin W' \cup W''$, and $s_1^i \in W' \cup W''$ imply that we have $N_G(W' \cup W'') \subseteq W_1^i$. By Lemma 2.7, no neighbor of W_1^i can be contained in $V - W' - W''$, i.e., $V - W' - W'' \subseteq W_1^i$ holds. This means $V = W' \cup W'' \cup W_1^i$, contradicting $N_G(Z_i) \neq \emptyset$ and $W_1^i \cup W' \cup W'' \subseteq Z_i$.

We next consider the case of $d(s_1^i) = 2$. Note that $|N_G(W_1^i)| = 1$, $N_G(W_1^i) \subseteq W_2^i$, and $Z_i = W_1^i \cup W_2^i$ hold. By $N_G(Z_i) \neq \emptyset$, $N_G(W_2^i) - Z_i \neq \emptyset$ holds. Assume by contradiction that there exist two distinct minimal deficient sets W' and W'' in $Z_i - \{u\}$ with respect to s_2^i. From $s_2^i \in W' \cap W''$ and the minimality of W' and W'', W' and W'' intersect each other. By $u \notin W' \cup W''$ and the connectedness of G, $N_G(W' \cup W'') \neq \emptyset$ holds. Lemma 2.8 implies that we have $|N_G(W' \cup W'')| = 1$. From the minimality of W_2, $W' - W_2 \neq \emptyset$ or $W'' - W_2 \neq \emptyset$ hold, and hence $(W' \cup W'') \cap W_1^i \neq \emptyset$ holds. The connectedness of $G[W_1^i]$, $u \notin W' \cup W''$, and $|N_G(W' \cup W'')| = 1$ imply that we have $N_G(W' \cup W'') \subseteq W_1^i$. Assume that $N_G(W' \cup W'') \subseteq W_1^i - W_2^i$ holds. Then the connectedness of $G[W_2^i]$ indicates that $W_2^i \subseteq W' \cup W''$ holds. Hence $N_G(W_2^i) \subseteq W' \cup W'' \cup N_G(W' \cup W'') \subseteq Z_i$, which contradicts that $N_G(W_2^i) - Z_i \neq \emptyset$ holds. Assume that $N_G(W' \cup W'') \subseteq W_1^i \cap W_2^i$ holds. By $|N_G(W' \cup W'')| = 1$ and Lemma 2.7, no neighbor of W_2^i can be contained in $V - W' - W''$, which contradicts that $W' \cup W'' \subseteq Z_i$ and $N_G(W_2^i) - Z_i \neq \emptyset$ hold. □

Proof of Lemma 3.8. Assume by contradiction that S^* is not a source set. Let W^* be a deficient set with $W^* \cap S^* = \emptyset$ such that no $W' \subset W^*$ satisfies this property (such W^* exists by Lemma 2.4). Let $w^* \in V$ be a vertex with $d(w^*) > |N_G(W^*)|$ such that W^* is a minimal deficient set with respect to w^*. Note that $S \cap W^* \neq \emptyset$ holds since S is a source set. Also note that each $W \in \mathcal{W}$ satisfies $S^* \cap W \neq \emptyset$ from the construction of S^* (note that for each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$, if $d(s_1^i) = 2$ and $u_i \in W_1^i - W_2^i$ hold, then $Z_i(u_i) \subseteq W_2^i$ holds by Lemma 3.7).

Let $W_1 \in \mathcal{W}$ be a deficient set with $W_1 \cap S = \{s_1\}$ and $s_1 \in W^*$ and $\mathcal{X}_1$ be a chain containing W_1 without loss of generality. Now we have $W^* - W_1 \neq \emptyset$ (resp., $W_1 - W^* \neq \emptyset$) since $W^* \subset W_1$ and $s_1 \in W^*$ would contradict the minimality of W_1 (resp., we have $S^* \cap (W_1 - W^*) \neq \emptyset$). Let $N_G(W_1) = \{x_1, y_1\}$ and $N_G(W^*) = \{x^*, y^*\}$, where $x_1 \in W^*$ and $x^* \in W_1$ (note that $W^* \cap N_G(W_1) \neq \emptyset \neq W_1 \cap N_G(W^*)$ holds by Lemma 2.5). Let $Z^* = W_1 \cup W^*$.

Claim A.1 *$V - \bigcup_{W \in \mathcal{W}} W - Z^* \neq \emptyset$ holds, or some $W \in \mathcal{W}$ is disjoint with Z^*. Hence, if a minimal deficient set $W \in \mathcal{W}$ intersects with Z^*, then $N_G(W) - Z^* \neq \emptyset$ holds.*

Proof. The case of $S_1 \neq \emptyset$ is trivial by $S_1 \subseteq V - \bigcup_{W \in \mathcal{W}} W - Z^*$. We consider the case of $|S| \geq 5$. Every $W \in \mathcal{W}$ satisfies $W - W^* \neq \emptyset$ by $(W - W^*) \cap S^* \neq \emptyset$. Since $|N_G(W^*)| \leq 2$ holds and no three sets W_i, W_j, and W_k in $\mathcal{W}$ satisfy $W_i \cap W_j \cap W_k \neq \emptyset$ by Lemma 3.3(ii), at most four sets in $\mathcal{W}$ has an intersection with W^*. $|S| \geq 5$ and this imply that some $W' \in \mathcal{W}$ is disjoint with Z^*. □

By this claim, we have $|N_G(Z^*)| > 0$. So one of the statements (i) – (iii) in Lemma 2.8 holds for W_1 and W^* by $s_1 \in W^*$; $|N_G(Z^*)| = 1$ holds. Let $\{z^*\} = N_G(Z^*)$. There are the following two possible cases (I) and (II). (I) Every $W \in \mathcal{W} - \{W_1\}$ satisfies $S \cap W \cap W^* = \emptyset$. (II) Some set $W \in \mathcal{W} - \{W_1\}$ satisfies $S \cap W \cap W^* \neq \emptyset$.

(I) From the assumption (I), we have $Z^* \cap S = \{s_1\}$. If $d(w^*) = 2$ would hold, then we have $|N_G(W^*)| \leq 1$ and $N_G(W^*) \subseteq W_1$, and Lemma 2.8 implies that $|N_G(W_1)| = 2$ and $d(s_1) = 3$ would hold, from which $|N_G(W^*)| \leq d(W^*) - 2$ holds, a contradiction to Lemma 2.6. Hence $d(w^*) = 3$ holds. Then $w^* \neq s_1$ would imply that $Z^* - \{s_1\}$ is a deficient set by $|N_G(Z^*)| = 1$, contradicting $|S \cap Z^*| = 1$ and the feasibility of S. Hence we have $w^* = s_1$. Therefore, $d(s_1) = 3$ and $|N_G(W^*)| = |N_G(W_1)| = 2$ hold.

Assume that $\mathcal{X}_1 = \{W_1\}$ holds. By Lemma 2.7 and $s_1 = w^*$, we have $(s_1, z^*) \in E$ or there are at least two vertex-disjoint paths between s_1 and z^* in both cases of $z^* = y_1$ and $z^* = y^*$. Hence, we can see $Z_1 = Z^*$ from the construction of Z_1 and $Z^* \cap S = \{s_1\}$. Moreover, $Z_1 \neq \emptyset$ and $S^* \cap W^* = \emptyset$ imply that $u_1 \in W_1 - W^*$ holds. Since W^* is a deficient set with $s_1 \in W^*$ and $u_1 \notin W^*$, we have $Z_1(u_1) \subseteq W^*$ from the minimality of $Z_1(u_1)$ and Lemma 3.7, which contradicts $S^* \cap W^* = \emptyset$.

Assume that $|\mathcal{X}_1| \geq 2$ holds. Let $W_2 \in \mathcal{W}$ satisfy $W_1 \cap W_2 \neq \emptyset$ and $\{s_2\} = S \cap W_2$. We have $s_2 \notin Z^*$ by $S \cap Z^* = \{s_1\}$, and $N_G(W_2) - Z^* \neq \emptyset$ by Claim A.1. Hence, by $|N_G(W_2) \cap W_1| = 1$ and Lemma 3.3(i), we have $N_G(W_2) \cap Z^* = N_G(W_2) \cap W_1 = \{s_1\}$. Moreover, Lemma 2.7 implies that $z^* = s_2$ holds, and hence $|N_G(Z^* \cup W_2)| = 1$ holds. By $z^* = s_2$ and $S \cap Z^* = \{s_1\}$, we see that each $W \in \mathcal{W} - \{W_1, W_2\}$ is disjoint with Z^*. This implies that $W_1^1 = W_1$, $W_2^1 = W_2$, and $Z_1 = Z^* \cup W_2$ hold. Hence we see $u_1 \in W_1 - W^*$. Since W^* is a deficient set with $s_1 \in W^*$ and $u_1 \notin W^*$, we see $Z_1(u_1) \subseteq W^*$ from Lemma 3.7, which contradicts $S^* \cap W^* = \emptyset$.

(II) Let $W_2 \in \mathcal{W} - \{W_1\}$ satisfy $S \cap W_2 \cap W^* = \{s_2\}$. We claim that

$$(W_1 - W_2) \cup (W_2 - W_1) \subseteq W^*, \text{ i.e., } W_2 \subseteq Z^* \quad (3)$$

holds. If $W_2 - Z^* \neq \emptyset$ would hold without loss of generality, then Claim A.1 implies that $N_G(W_2) - Z^* \neq \emptyset$ would hold, which contradicts $s_2 \in W^*$, $|N_G(Z^*)| = 1$, and Lemma 2.7. So we have $W_2 \subseteq Z^*$ and $W_1 \subseteq W_2 \cup W^*$ similarly. By (3) and $W_1 - W^* \neq \emptyset \neq W_2 - W^*$, we have $W_1 \cap W_2 \neq \emptyset$. By $W_1 \cap S^* \neq \emptyset \neq W_2 \cap S^*$ and $W^* \cap S^* = \emptyset$, we have $S^* \cap Z^* = S^* \cap W_1 \cap W_2$. Moreover, we can see that any $W \in \mathcal{W} - \{W_1, W_2\}$ satisfies $W \cap S \cap W^* = \emptyset$, since if there exists $W_3 \in \mathcal{W} - \{W_1, W_2\}$ with $W_3 \cap S \cap W^* \neq \emptyset$, then a vertex in $W_3 - W^*$ is also contained in $W_1 \cap W_2$, contradicting Lemma 3.3(ii).

Assume that $\mathcal{X}_1$ is of type (C) and $W_1 = W_1^1$ and $W_2 = W_2^1$ hold without loss of generality. $u_1 \in W_1 - W^*$ holds. We have $W_1 \cup W_2 \subseteq Z_1 \subseteq Z^*$ since $|N_G(Z_1)| = 1$ and $N_G(Z_1) \subseteq N_G(W_2)$ hold from the definition of Z_1. Let $N_G(Z_1) \cap N_G(W_2) = \{y_2\}$. Assume $y^* \notin Z_1$ holds. Lemma 2.8 implies that we have $x^* \in W_1 \cap W_2$. Note that $N_G(Z_1 \cap W^*) = \{x^*, y_2\}$, $|N_G(W_2)| = 2$, and $d(s_2) = 3$ hold by (3). Then in both cases of $d(s_1) = 2$ and $d(s_1) = 3$, $Z_1 \cap W^*$ is a deficient set by $\{s_1, s_2\} \subseteq Z_1 \cap W^*$ and so we have $Z_1(u_1) \subseteq Z_1 \cap W^*$ from Lemma 3.7, contradicting $S^* \cap W^* = \emptyset$. Assume that $y^* \in Z_1$ holds. By (3), we have $N_G(W^*) \subseteq W_1 \cap W_2$, $N_G(W_1 - W_2) \subseteq W_2$, and $N_G(W_2 - W_1) \subseteq W_1$. This means that $(N_G(W_1) - W_2) \cup (N_G(W_2) - W_1) \subseteq N_G(W_1 \cap W_2)$ holds. Hence $|N_G(W_1)| = |N_G(W_2)| = 2$ and $|N_G(W_1 \cup W_2)| = 1$ hold. By Claim A.1 and this, we have $W^* \subseteq W_1 \cup W_2$. From the construction of Z_1, we have $Z_1 = W_1 \cup W_2$. Now $u_1 \in W_1 - W^*$ holds, and W^* is a deficient set with $W^* \subseteq Z_1 - \{u_1\}$. Hence $Z_1(u_1) \subseteq W^*$ holds, contradicting $S^* \cap W^* = \emptyset$.

Assume that $\mathcal{X}_1$ is not a chain of type (C) or $\mathcal{X}_1$ is a chain of type (C) with $\{W_1, W_2\} \neq \{W_1^1, W_2^1\}$. Then from the definition of chains and Lemma 3.3(i), we have $N_G(W_1) - W_2 \neq \emptyset \neq N_G(W_2) - W_1$, $N_G(W_2) \cap W_1 = \{s_1\}$, $N_G(W_1) \cap W_2 = \{s_2\}$, and $d(s_1) = d(s_2) = 3$. We claim that $\mathcal{X}_1 = \{W_1, W_2\}$ holds. If there exists a deficient set $W_3 \in \mathcal{W} - \{W_1, W_2\}$ with $W_2 \cap W_3 \neq \emptyset$ without loss of generality, then $S \cap W_3 \cap Z^* = \emptyset$ and Claim A.1 imply that $N_G(W_3) - Z^* \neq \emptyset$ would hold and hence $\mathcal{X}_1$ would be of type (C) with $\{W_1, W_2\} = \{W_1^1, W_2^1\}$, a contradiction. Let $\{y_i\} = N_G(W_i) - W_j$ hold for $\{i, j\} = \{1, 2\}$. Since $\mathcal{X}_1$ is not of type (C), we have $y_1 \neq y_2$ and $y_1 \neq z^* \neq y_2$. Hence $z^* \in N_G(W^*)$ holds and Lemma 2.8 implies that $x^* \in W_1 \cap W_2$ holds. Let Z' be a vertex set in Z^* with $W_1 \cup W_2 \subseteq Z'$ and $|N_G(Z')| = 1$ such that no $Z'' \subset Z'$ satisfies this property (such Z' exists by $|N_G(Z^*)| = 1$ and $W_1 \cup W_2 \subseteq Z^*$). Then we see that $Z' = Z_1$ holds. Now note that $Z^* - W^* \subseteq W_1 \cap W_2$ and $N_G(W^*) \cap (W_1 \cup W_2) = \{x^*\}$ hold by (3) and $x^* \in W_1 \cap W_2$. Hence, $N_G(Z_1 \cap W^*) = \{z_1, x^*\}$ holds, where $\{z_1\} = N_G(Z_1)$, and $Z_1 \cap W^*$ is a deficient set by $\{s_1, s_2\} \subseteq Z_1 \cap W^*$. We have $u_1 \in Z^* - W^* \subseteq W_1 \cap W_2$ by $Z_1 \neq \emptyset$ and $W^* \cap S^* = \emptyset$, and hence $Z_1(u_1) \subseteq Z_1 \cap W^*$ holds, contradicting $S^* \cap W^* = \emptyset$. □

Proof of Lemma 3.9. Let $\{s_i\} = S \cap W_i$ and $\{s\} = S \cap W$. Since $|S| \geq 3$ or $V \neq \bigcup_{W \in \mathcal{W}} W$ hold, we have $N_G(W) - W_i \neq \emptyset$, from which and $|N_G(W)| \leq 2$, $|N_G(W) \cap W_i| = 1$ holds. Lemma 3.3(i) says that $\{s_i\} = N_G(W) \cap W_i$ holds. Note that by $|W_i| \geq 2$ and Lemma 2.6, we have $|N_G(W_i)| = 2$. Hence, $N_G(W_i) \cap (W - \{s\}) \neq \emptyset$ holds. By this, Lemma 2.7, $s \notin X$, and $|N_G(X)| = 1$, we see that $X \cap W \neq \emptyset$, $N_G(X) = \{s\}$, and $N_G(W_i) \subseteq X \cup N_G(X)$ hold; $N_G(X \cap W) = \{s, s_i\}$ and $N_G((X \cap W) \cup W_i) = \{s\}$ hold and there exist at least two vertex-disjoint paths between s and s_i in $G[(X \cap W) \cup \{s, s_i\}]$. Therefore, any minimal deficient set $W' \subseteq (X \cap W) \cup (W_i - \{v\})$ with respect to s_i satisfies $X \cap W \cap W' \neq \emptyset$ for a vertex $v \in W_i - \{s_i\}$ (note that $(X \cap W) \cup (W_i - \{v\})$ is a deficient set by $|N_G((X \cap W) \cup W_i)| = 1$ and $d(s_i) = d(W_i) = 3$). By $X \cap W \cap S = \emptyset$, such W' has property (P) with respect to S. It is not difficult to see that such W' can be found in $O(|W_i \cup W|)$ time by computing a biconnected component containing s_i in $G[W_i \cup W]$. □

Proof of Lemma 3.12. From the definition of chains and $Z_i(u) \subseteq Z_i$, it suffices to show that we have $Z_i(u) \cap W_3^i = \emptyset$ for a chain $\mathcal{X}_i$ with $|\mathcal{X}_i| \geq 3$ and $d(s_1^i) = 3$ and a vertex $u \in W_1^i$. We have $N_G(W_3^i) - W_2^i \neq \emptyset$ since otherwise $\mathcal{W} = \{W_1, W_2, W_3\}$ holds, contradicting that $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$ hold. Hence $|N_G(W_3^i) \cap W_2^i| = 1$ holds. By Lemma 3.3(i), we have $\{s_2^i\} = N_G(W_3^i) \cap W_2^i$. This implies that $Z' = Z_i - W_3^i - \{s_2^i\}$ satisfies $N_G(Z') = \{s_2^i\}$. Now, by $W_1^i \cap W_3^i = \emptyset$ and $u \in W_1^i - \{s_1^i\}$, we see $N_G(Z' - \{u\}) = \{u, s_2^i\}$ and $s_1^i \in Z' - \{u\}$. This implies that $Z_i(u) \subseteq Z' - \{u\} \subseteq V - W_3^i$ holds. □