The Symmetry Ratio of a Network

Anthony H. Dekker

Defence Systems Analysis Division
Defence Science and Technology Organisation
Department of Defence, Canberra, Australia
Email: `dekker@ACM.org`

Bernard Colbert

School of Information Technology
Deakin University
Burwood, Victoria, Australia

Abstract

In this paper, we introduce a new measure of symmetry which we call the *symmetry ratio* of a network, defined to be the ratio of the number of distinct eigenvalues of the network to $D+1$, where D is the diameter. The symmetry ratio has utility in partially predicting the robustness of a network in the face of attack. We prove a number of results placing bounds on the symmetry ratio for several families of networks, including distance-transitive networks, prisms, twisted prisms, antiprisms, tori, Cayley graphs, and random graphs.

Keywords: Communications network, robustness, graph theory, graph spectrum, symmetry.

1 Introduction

In recent work, we studied the robustness of various critical infrastructure network topologies under multiple terrorist attacks (Dekker 2005). We modelled these networks as undirected graphs with limited link capacity, carrying simulated traffic. Each attack removed the most important (most *central*) remaining node, thus causing traffic to be re-routed. We found that networks began to fail when the number of attacks α was equal to the *node connectivity* κ (the smallest number of node-independent paths between pairs of nodes). The severity of failure was related to the *average degree* d_{ave}. In particular, the performance (percentage of messages successfully received) could be predicted as a function of $\log d_{\text{ave}}$ and of the difference $\alpha - \kappa$ between node connectivity and number of attacks (we call this difference the *relative attack count*).

The vertical axis of the scatter-plot in Figure 1 is the difference between actual and predicted values of (the logarithm of) performance for numbers of attacks α ranging from 1 to 6, and 61 different sixty-node networks (some designed, and some randomly generated). Each network corresponds to a column of six squares. The different colours represent different networks, and the numbers in each square are the different values of α. Twelve of the networks used are highlighted. The variation that remains after taking d_{ave} and $\alpha - \kappa$ into account is related to *symmetry* (which is the horizontal axis in Figure 1).

In general, symmetrical networks performed much better than random networks. In particular, the Rhombicosidodecahedron in Figure 1(a) and the Snub Dodecahedron in Figure 1(c) could absorb 6 attacks without failing. However, the Ring in Figure 1(f) and the Prism in Figure 1(i) performed poorly, for reasons discussed in Dekker & Colbert (2004*a*).

Graph	No. of Autos	Symmetry Category	r
Ring	120	1	1
Soccer Ball	120	3	1.5
Random Tree	1.5×10^{29}	5	1.86
6×10 Torus	240	3	2.11
Figure 2	2	5	4.52
2-linked SF	1	5	10

Table 1: Some Candidate Symmetry Measures

The superior performance of the symmetrical networks over the random networks leads us to ask: *how can symmetry be quantified?*

One obvious answer is to count the number of *automorphisms* of a network (i.e. the size of the automorphism group). Most random networks have only the trivial automorphism (Theorem 9.3 of Bollobás (2001)), while, for example, the Rhombicosidodecahedron in Figure 1(a) has 120 automorphisms. However, trees, even randomly generated ones, have many more automorphisms, e.g. 1.5×10^{29} for the random tree in Figure 1(j). This approach therefore greatly exaggerates the symmetry of trees, and does not sufficiently distinguish other graphs.

Another obvious method is to use standard symmetry categories, and to rank the symmetry of a network as, for example:

1. for distance-transitive networks, e.g. the ring in Figure 1(f).
2. for symmetric non-distance-transitive networks.
3. for node-similar (i.e. vertex-transitive) non-symmetric networks, including the networks highlighted in Figure 1 except 1(b), 1(d), 1(f), and 1(j).
4. for regular non-node-similar networks.
5. for all other networks, including those in Figures 1(b), 1(d), and 1(j).

However, this approach does not differentiate between the partial symmetry of trees, and the lack of symmetry of random networks. For the purpose of assessing real-world networks (such as the one in Figure 2), there are too many categories at the symmetrical end of the scale, and not enough at the non-symmetrical end.

In this paper, we introduce an easily calculated measure of symmetry which we call the *symmetry ratio* r of a network, and define it to be:

$$r = \frac{\epsilon}{D+1}$$

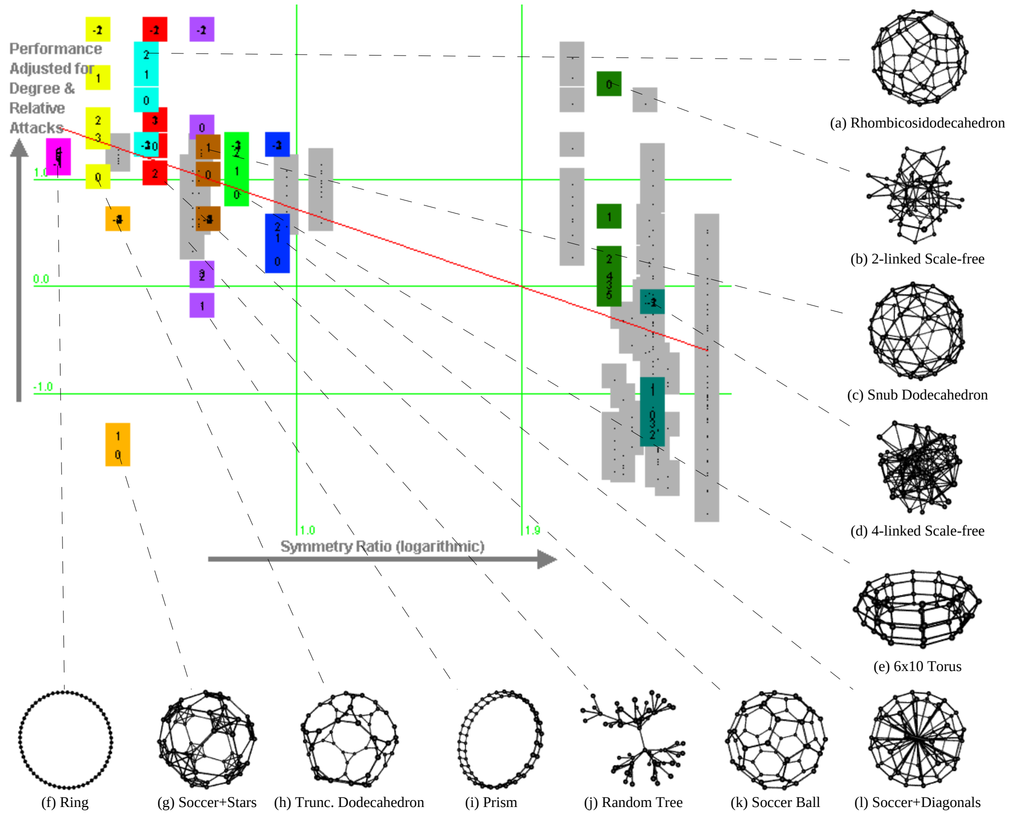


Figure 1: Symmetry Ratio and Performance of Various Networks under Attack

where ϵ is the number of distinct *eigenvalues* of the network, and D is the *diameter*. The horizontal axis of the scatter-plot in Figure 1 shows the (logarithm of the) symmetry ratio, which increases as networks become less symmetrical. Table 1 compares the symmetry ratio r for some of the networks in Figure 1 (and the network in Figure 2) against the two less useful candidate symmetry measures we have discussed (number of automorphisms and standard symmetry category).

Figure 1 shows a statistically extremely significant effect of the symmetry ratio on performance in the face of attack, and taking the symmetry ratio into account improves our earlier prediction of performance from 79% of variance to 84% of variance. This confirms that the symmetry ratio is indeed of practical utility.

Figure 2 shows a real-world telecommunications network (the Qwest Internet Backbone Network), redrawn from Dodge (2004). This network has a symmetry ratio of 4.52, which would fall between the high-symmetry networks on the left of Figure 1 ($1 \leq r \leq 3$), and the low-symmetry networks on the right of Figure 1 ($8.5 \leq r \leq 15$), as we might expect.

In the body of this paper we prove a number of results placing bounds on the symmetry ratio for several families of networks (summarised in Table 4), including distance-transitive networks, prisms, and Cayley graphs.

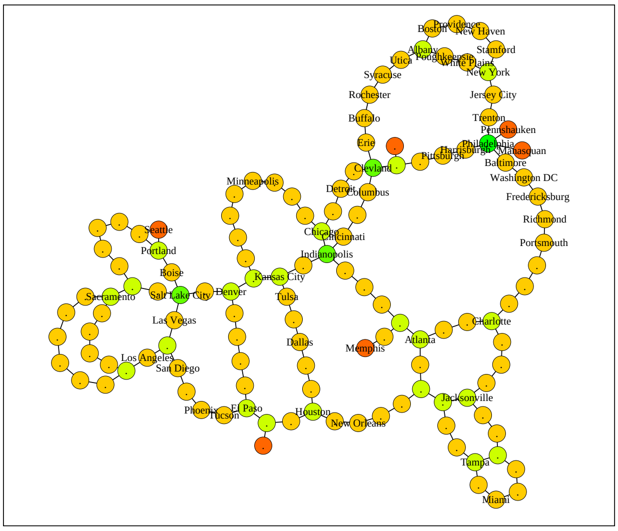


Figure 2: A Real-World Telecommunications Network with $r = 4.52$

2 Basic Concepts

Definition 2.1 We model a network as an (undirected) *graph*, consisting of *nodes* and *links*. We restrict our attention to *connected* graphs, and define:

(i) If a graph has n nodes, then we say that the graph has *size* n.

(ii) If a node has d outgoing links, then we say that the node has *degree* d.

(iii) If every node of a graph has the same degree, then we say that the graph is *regular*; in this case we also speak of the degree d of the graph.

(iv) An *automorphism* of a graph is a permutation π of the nodes which preserves links, i.e. a—b is a link if and only if πa—πb is a link.

(v) A graph is *node-similar* (more usually, *vertex-transitive*) if for any two nodes a and b there is an automorphism π such that $\pi a = b$.

(vi) A graph is *symmetric* if for any two links a—b and x—y there is an automorphism π such that $\pi a = x$ and $\pi b = y$.

(vii) The distance $\delta(x, y)$ between two nodes x and y is the number of links on the shortest path between them.

(viii) The *diameter* D of the graph is the largest value of $\delta(x, y)$.

(ix) A graph is *distance-transitive* if for all nodes a, b, x, and y such that $\delta(a, b) = \delta(x, y)$, there is an automorphism π such that $\pi a = x$ and $\pi b = y$.

The distance-transitive graphs include the completely connected graphs K_n, rings, hypercubes, hamming graphs (Dekker & Colbert 2004*a*), and Platonic polyhedra (the tetrahedron, cube, octahedron, dodecahedron, and the icosahedron in Figure 4(d)).

The *spectrum* of a graph is the set of *eigenvalues* of the adjacency matrix of the graph. We write the spectrum in the form:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_\epsilon \\ n_1 & n_2 & \dots & n_\epsilon \end{pmatrix}$$

where n_i is the *multiplicity* of the eigenvalue λ_i. Clearly $\sum_i n_i = n$, where n is the size of the graph.

Definition 2.2 In order to ensure $r \geq 1$ (see Proposition 2.3), we define the *symmetry ratio* r of a graph by:

$$r = \frac{\epsilon}{D + 1}$$

where ϵ is the number of distinct eigenvalues of the graph, and D is the *diameter*.

Proposition 2.3 *For any graph of size* $n \geq 3$*:*

(i) $D + 1 \leq \epsilon \leq n$.

(ii) $1 \leq r \leq \frac{n}{3}$.

(iii) $D \geq \lceil (\log(n-1))/(\log d) \rceil$, *for regular graphs of degree* d.

(iv) $r \leq n/(1 + \lceil (\log(n-1))/(\log d) \rceil)$, *for regular graphs of degree* d.

Proof.

(i) By Corollary 2.7 of Biggs (1993).

(ii) From (i), and the fact that the completely connected graph K_n has $\epsilon = 2$ and $D = 1$, so $r = 1$, but otherwise $D \geq 2$, and so $r = \epsilon/(D+1) \leq n/3$.

(iii) From the Moore bound for $d \geq 3$:

$$n \leq \frac{d(d-1)^D - 2}{d - 2}$$

proved in Theorem 10.1 of Bollobás (2001), we can derive $n - 1 \leq d^D$, and the result follows. For $d = 2$ the graph is a ring and $D = \lfloor \frac{n}{2} \rfloor \geq \lceil (\log(n-1))/(\log d) \rceil$.

(iv) From (iii). □

Definition 2.4 The simplest families of graphs we consider are the following:

(i) The *path graph* P_n has n nodes connected in a straight line (and hence diameter $n - 1$).

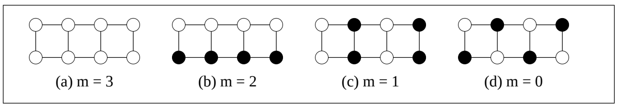


Figure 3: Possible Eigenvectors for Simple Eigenvalues of the Prism

(ii) The *star tree* T_n has $n \geq 3$ nodes $0, \dots, n-1$, with the only links from node 0 to the other nodes.

Proposition 2.5 *For the path graph* P_n, $\epsilon = n$ *and* $r = 1$.

Proof. Since $D = n-1$ and $\epsilon \geq D+1$ by Proposition 2.3(i). □

Proposition 2.6 *For the star tree* T_n, $\epsilon = 3$ *and* $r = 1$.

Proof. Clearly $D = 2$, and the result follows since the spectrum is:

$$\begin{pmatrix} \sqrt{n-1} & 0 & -\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}$$

□

The following important class of graphs includes the Peterson graph (see Figure 6.14 of Gibbons (1985)):

Definition 2.7 A *strongly regular* (n, d, ν, μ) graph has n nodes, and is regular with degree d, such that any two adjacent nodes have $\nu \geq 0$ common neighbours, and and two non-adjacent nodes have $\mu \geq 1$ common neighbours.

Proposition 2.8 *For any strongly regular graph,* $\epsilon = 3$ *and* $r = 1$.

Proof. Clearly $D = 2$. By Note 3d of Biggs (1993), $\epsilon = 3$, and so $r = 1$. □

Proposition 2.9 *For any distance-transitive graph,* $r = 1$.

Proof. By Theorem 20.7 of Biggs (1993), $\epsilon = D+1$. □

In fact, the result of Biggs (1993) is proved for distance-regular graphs, a class which includes both the strongly regular and distance-transitive graphs.

3 The Prism

Prism graphs, such as the ones shown in Figure 1(i) and Figure 4(a), have degree 3, and a high degree of symmetry. In fact, $r < 2$:

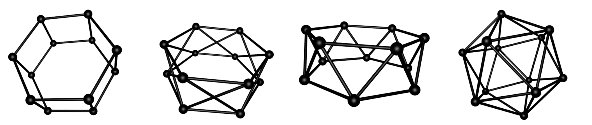
Theorem 3.1 *Consider a prism with* $2n$ *nodes. Then:*

(i) If n *is even,* $\epsilon \leq n + 2$.

(ii) If n *is odd,* $\epsilon \leq n + 1$.

(iii) $r \leq 2\left(\lfloor \frac{n}{2} \rfloor + 1\right) / \left(\lfloor \frac{n}{2} \rfloor + 2\right) < 2$.

Proof. Let λ be an eigenvalue of multiplicity 1, and $\mathbf{x}$ a corresponding real eigenvector. Let $\mathbf{P}$ be a permutation matrix representing an automorphism π of the prism. By Lemma 15.3 of Biggs (1993), $\mathbf{Px} = \mathbf{x}$ or $\mathbf{Px} = -\mathbf{x}$, and the entries in $\mathbf{x}$ differ only in sign, since the prism is node-similar (vertex-transitive).



(a) Prism (b) Twisted Prism (c) Antiprism (d) Icosahedron

Figure 4: Some Twelve-Node Graphs

The possible eigenvectors $\mathbf{x}$ are then as shown in Figure 3, where open circles denote nodes corresponding to positive elements of $\mathbf{x}$, and black circles denote nodes corresponding to negative elements. Cases (c) and (d) in Figure 3 are possible only if n is even. The cases can be distinguished by the number m of nodes adjacent to a given node which have the same corresponding entry x_i in $\mathbf{x}$. If $\mathbf{A}$ is the adjacency matrix of the prism:

$$(\mathbf{Ax})_i = mx_i - (3-m)x_i = (2m-3)x_i = \lambda x_i$$

Hence if n is even, λ is one of -3, -1, 1, or 3, while if n is odd, λ is 1 or 3. All other eigenvalues have multiplicity at least 2, and so it follows that $\epsilon \le (2n-4)/2+4 = n+2$ if n is even, and $\epsilon \le (2n-2)/2+2 = n+1$ if n is odd, i.e. $\epsilon \le 2\lfloor \frac{n}{2} \rfloor + 2$.

The diameter of the prism is $1 + \lfloor \frac{n}{2} \rfloor$, and so (iii) follows. □

The bound here is tight: for the pentagonal prism ($n = 5$) with spectrum:

$$\begin{pmatrix} 3 & \frac{1+\sqrt{5}}{2} & 1 & \frac{-3+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{-3-\sqrt{5}}{2} \\ 1 & 2 & 1 & 2 & 2 & 2 \end{pmatrix}$$

$\epsilon = 6 = n+1$ and $D = 3$, so $r = 2(2+1)/(2+2) = 1.5$. For the decagonal prism ($n = 10$), $\epsilon = 12 = n+2$ and $D = 6$, so $r = 2(5+1)/(5+2) = \frac{12}{7}$.

Analogous results can be proved for other node-similar graphs. Note that the cube is also a prism, but (since it is distance-transitive) has the minimum value of $r = 1$.

4 The Twisted Prism

A twisted prism resembles a prism, but each face is replaced by a pair of crossing links, so that the graph has degree 4, as shown in Figure 4(b).

Theorem 4.1 *Consider a twisted prism with* $2n$ *nodes. Then:*

(i) $\epsilon \le 2 + \lfloor \frac{n}{2} \rfloor$.

(ii) $r \le 1 + 1 \,/\, \left(1 + \lfloor \frac{n}{2} \rfloor\right) \le \frac{4}{3}$.

Proof.

(i) The graph consists of n pairs of nodes, each with the same set of neighbours, and hence the eigenvalue 0 has multiplicity n. Also, by Proposition 16.7 of Biggs (1993), since the twisted prism is symmetric, the only eigenvalues of multiplicity 1 are 4 and -4 (and only 4 if n is odd, i.e. the graph is not bipartite). Hence $\epsilon \le 3+(n-2)/2 = 2+\lfloor \frac{n}{2} \rfloor$ if n is even, and $\epsilon \le 2+(n-1)/2 = 2+\lfloor \frac{n}{2} \rfloor$ if n is odd.

(ii) For $n \ge 4$, the diameter $D = \lfloor \frac{n}{2} \rfloor$, and the result follows, whereas for $n = 3$ the graph is an octahedron and $r = 1$. □

The bound in this theorem is tight: for the pentagonal twisted prism ($n = 5$), $\epsilon = 4 = 2 + \lfloor \frac{n}{2} \rfloor$ and $D = 2$, so $r = \frac{4}{3}$. For the hexagonal twisted prism ($n = 6$), $\epsilon = 5 = 2 + \lfloor \frac{n}{2} \rfloor$ and $D = 3$, so $r = 1.25$.

5 Cayley Graphs

Given a group G, and a set S of elements of G, we say that G is *generated by* S if the elements of G can all be built up by using the group binary operation xy, the group inverse x^{-1}, the group identity 1, and the elements of S.

Definition 5.1 If the group G is generated by S, we define:

(i) The *closure* of S is $\widehat{S} = S \cup \{s^{-1} \,|\, s \in S\} \setminus \{1\}$.

(ii) The *Cayley graph* $\Gamma(G, S)$ is the graph whose nodes are the elements of G, and whose links are x — sx for every $x \in G$ and $s \in S$.

For example, the group $\mathbb{Z}_5$ is generated by the set $\{2\}$, and the corresponding Cayley graph is a pentagon. There are usually multiple Cayley graphs for a given group G, depending on the choice of S. Conversely, different groups may have the same Cayley graphs.

Proposition 5.2 *Let* G *be generated by* S*, and let* $\Gamma(G, S)$ *be the corresponding Cayley graph. Then:*

(i) $\Gamma(G, S)$ *is regular and node-similar (vertex-transitive).*

(ii) $\Gamma(G, S)$ *has degree* $|\widehat{S}|$.

Proof.

(i) Proposition 16.2 of Biggs (1993).

(ii) Considering possible links x — sx and $s^{-1}y$ — y. □

6 The Antiprism

Antiprism graphs, such as the one shown in Figure 4(c), have degree 4, and are formed from two rings, connected by a ring of triangles facing alternately up and down.

Theorem 6.1 *Consider an antiprism with* $2n$ *nodes. Then:*

(i) $\epsilon \le n + 1$.

(ii) $r \le (n+1) \,/\, \left(\lfloor \frac{n+1}{2} \rfloor + 1\right) < 2$

Proof.

(i) The antiprism is in fact a Cayley graph for $\mathbb{Z}_{2n}$, with even numbers on one ring, and odd numbers on the other, generated by e.g. $\{1, 2\}$. Consequently, it is a *circulant* graph, with each row of the adjacency matrix $\mathbf{A}$ being simply the top row shifted. By Proposition 3.5 of Biggs (1993), the eigenvalues λ_i for $i = 0 \ldots 2n-1$ are given by:

$$\begin{aligned} \lambda_i &= \sum_{j=1}^{2n-1} \mathbf{A}_{0j} \cos\left(\frac{\pi i j}{n}\right) \\ &= \cos\left(\frac{\pi i}{n}\right) + \cos\left(\frac{2\pi i}{n}\right) \\ &\quad + \cos\left(\frac{(2n-2)\pi i}{n}\right) + \cos\left(\frac{(2n-1)\pi i}{n}\right) \\ &= 2\cos\left(\frac{\pi i}{n}\right) + 2\cos\left(\frac{2\pi i}{n}\right) \end{aligned}$$

So $\lambda_0 = 4\cos 0 = 4$, $\lambda_n = 2\cos\pi + 2\cos(2\pi) = 0$, and similarly $\lambda_{2n-i} = \lambda_i$ for $1 \le i \le n-1$. Hence there are at most $n+1$ distinct eigenvalues.

(ii) The diameter $D = \left\lfloor \frac{n+1}{2} \right\rfloor$, and the result follows. □

The bound in this theorem is tight: for the 4-sided antiprism, $\epsilon = 5 = 4+1$ and $D = 2$, so $r = \frac{5}{3}$. For the 7-sided antiprism, $\epsilon = 8 = 7+1$ and $D = 4$, so $r = 1.6$. The method of Theorem 3.1 provides an alternative proof here.

Analogous results can be proved for other Cayley graphs of $\mathbb{Z}_n$. Note that the octahedron is also an antiprism, but (since it is distance-transitive) has the minimum value of $r = 1$.

7 Group Representations

Each group G can have multiple *representations* as complex matrices, where the group binary operator is implemented by matrix multiplication (i.e. the representation is a homomorphism). The *character* of a representation is the function that maps each element of G to the sum of diagonal elements of the corresponding matrix. The *irreducible characters* of a group are the characters of irreducible representations.

A *conjugacy class* of a group is a set of elements closed under conjugation, i.e. if $c \in C$ and $g \in G$, then $gcg^{-1} \in C$.

The number of irreducible characters is equal to the number of conjugacy classes (Theorem 15.3 of James & Liebeck (1993)). For each group we can therefore form a square *character table* with (representatives of) conjugacy classes as columns, irreducible characters as rows, and values of the irreducible characters on the conjugacy classes as elements (these values are the same irrespective of which representative of a conjugacy class is chosen). For example, the group D_3 is generated by $\{a, b\}$ with $a^3 = b^2 = 1$ and $ab = ba^2$, and has three conjugacy classes: $\{1\}$, $\{a, a^2\}$, and $\{b, ba, ba^2\}$. The character table is:

D_3	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Each group has at least one irreducible character with $\chi(1) = 1$, while for abelian groups, every irreducible character has $\chi(1) = 1$. For every group, $\sum_i (\chi_i(1))^2 = |G|$. In the case of D_3, $1 + 1 + 2^2 = 6$. Many software packages for group theory will generate character tables of groups.

For Cayley graphs, there is a close relationship between the character tables and the eigenvalues. Since $\chi(1) = 1$ for every irreducible character of an abelian group, we consider abelian groups separately:

Proposition 7.1 *Let G be a finite abelian group generated by S. Then the eigenvalues of $\Gamma(G, S)$ correspond to the irreducible characters of G, and are given by:*

$$\lambda_\chi = \sum_{s \in \widehat{S}} \chi(s)$$

Proof. By Theorem 6 of Murty (2003) or Corollary 3.2 of Babai (1979). □

Proposition 7.2 *Let G be a finite non-abelian group generated by S, with irreducible characters $\chi_1, \ldots, \chi_c$. Then the number of distinct eigenvalues of $\Gamma(G, S)$ satisfies $\epsilon \le \sum_i \chi_i(1)$, where the $\chi_i(1)$ are positive integers.*

Proof. By Note 16h of Biggs (1993) or Theorem 3.1 of Babai (1979). □

For example, the triangular prism is a Cayley graph of D_3 with spectrum:

$$\begin{pmatrix} 3 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

and $\sum_i \chi_i(1) = 1+1+2 = 4 = \epsilon$. As remarked above, $\sum_i (\chi_i(1))^2 = |G| = 6$.

If the number of conjugacy classes c is known, the bound on ϵ may not need knowledge of the character table, since $|G|$ will be the sum of c squares, one of which is 1, and in many cases this is uniquely defined. For example, with $c \le 4$ and $|G| \le 50$, there are only 4 possible non-unique sums of squares:

$$\begin{aligned} 28 &= 1+1+1+5^2 = 1+3^2+3^2+3^2 \\ 34 &= 1+2^2+2^2+5^2 = 1+1+4^2+4^2 \\ 39 &= 1+1+1+6^2 = 1+2^2+3^2+5^2 \\ 42 &= 1+1+2^2+6^2 = 1+3^2+4^2+4^2 \end{aligned}$$

8 The Torus

We can use group representation theory to bound ϵ for a torus such as the one shown in Figure 1(e), using the following result:

Proposition 8.1 *For abelian groups P and Q, each eigenvalue λ_χ of $P \times Q$ has the form $\lambda_\chi = \lambda_{\chi_P} + \lambda_{\chi_Q}$ for some eigenvalues λ_{χ_P} of P and λ_{χ_Q} of Q.*

Proof. If P is generated by A and Q is generated by B, then $P \times Q$ is generated by $\{(a, 1_Q) | \, a \in A\} \cup \{(1_P, b) | \, b \in B\}$, and so by Proposition 7.1:

$$\begin{aligned} \lambda_\chi &= \sum_{s \in \widehat{S}} \chi(s) = \sum_{a \in \widehat{A}} \chi(a, 1_Q) + \sum_{b \in \widehat{B}} \chi(1_P, b) \\ &= \sum_{a \in \widehat{A}} \chi_P(a)\chi_Q(1_Q) + \sum_{b \in \widehat{B}} \chi_P(1_P)\chi_Q(b) \end{aligned}$$

since irreducible representations of a product group are direct products of the individual representations, i.e. every irreducible character χ of $P \times Q$ can be expressed as a product of irreducible characters χ_P of P and χ_Q of Q (Theorem 19.18 of James & Liebeck (1993)). But $\chi_P(1_P) = \chi_Q(1_Q) = 1$ since P and Q are abelian, and so:

$$\lambda_\chi = \sum_{a \in \widehat{A}} \chi_P(a) + \sum_{b \in \widehat{B}} \chi_Q(b) = \lambda_{\chi_P} + \lambda_{\chi_Q}$$

□

Corollary 8.2 *For an $m \times n$ torus graph,*

(i) $\epsilon \le \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$

(ii) $r \le 1 + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor / \left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1\right)$

Graph	Figure 1 Reference	Generators	ϵ	D	r	Bound on r	Planar Bound
Rhombicosidodecahedron	(a)	$\{a, ab\}$	13	8	1.444	4	1.778
Snub Dodecahedron	(c)	$\{a, ab, b\}$	15	7	1.875	4	2
Soccer+Stars	(g)	$\{a, a^2, b\}$	9	6	1.286	4	—
Trunc. Dodecahedron	(h)	$\{ab, b\}$	13	10	1.182	3.2	1.6
Soccer Ball	(k)	$\{a, b\}$	15	9	1.5	3.2	1.6
Soccer+Diagonals	(l)	$\{a, b, a^2ba^3ba^2b\}$	15	5	2.5	4	—

Table 2: Some Cayley Graphs of A_5

Graph	Generators	ϵ	D	r	Bound on r	Planar Bound
Truncated Cube	$\{a, b\}$	8	6	1.143	2.5	1.429
Truncated Octahedron	$\{ab, b\}$	10	6	1.429	2.5	1.429
Small Rhombicuboctahedron	$\{a, ab\}$	8	5	1.333	2.5	1.667
Snub Cube	$\{a, ab, b\}$	9	4	1.8	3.333	2

Table 3: Some Cayley Graphs of S_4

Proof. Since the torus is a product of an m-ring (with $\epsilon = \lfloor \frac{m}{2} \rfloor + 1$ and $D = \lfloor \frac{m}{2} \rfloor$) and an n-ring (with $\epsilon = \lfloor \frac{n}{2} \rfloor + 1$ and $D = \lfloor \frac{n}{2} \rfloor$). □

This result generalises to a hypertorus (Dekker & Colbert 2004a) in the obvious way. The bound is tight: for the 6×5 torus, $\lfloor \frac{m}{2} \rfloor = 3$, $\lfloor \frac{n}{2} \rfloor = 2$, $\epsilon = 12$, $D = 5$, and $r = 2$. The method of Theorem 3.1 could also be applied, but produces a looser bound.

For the square ($n \times n$) torus, we can provide even tighter bounds:

Corollary 8.3 *Consider an $n \times n$ square torus. Then:*

(i) $\epsilon \le 1 + \frac{1}{2}\lfloor \frac{n}{2} \rfloor^2 + \frac{3}{2}\lfloor \frac{n}{2} \rfloor$, *if n is odd.*

(ii) $r \le 1 + \left(\lfloor \frac{n}{2} \rfloor^2 - \lfloor \frac{n}{2} \rfloor\right) / \left(2 + 4\lfloor \frac{n}{2} \rfloor\right)$, *if n is odd.*

(iii) $\epsilon \le 1 + \frac{1}{2}\lfloor \frac{n}{2} \rfloor^2 + \frac{3}{2}\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{4} \rfloor$, *if n is even.*

(iv) $r \le 1 + \left(\lfloor \frac{n}{2} \rfloor^2 - \lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{n}{4} \rfloor\right) / \left(2 + 4\lfloor \frac{n}{2} \rfloor\right)$, *if n is even.*

Proof.

(i) We have $\lambda_\chi = \lambda_i + \lambda_j$, where:

$$\lambda_i = \sum_{k=1}^{n-1} \mathbf{A}_{0k} \cos\left(\frac{2\pi ik}{n}\right) = 2\cos\left(\frac{2\pi i}{n}\right)$$

i.e. $\lambda_0 = 2$ and $\lambda_{n-i} = \lambda_i$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$. Then consider sums $\lambda_i + \lambda_j$.

(ii) Since $D = 2\lfloor \frac{n}{2} \rfloor$.

(iii) Similar to (i), but the even ring is bipartite, and hence its spectrum is symmetrical about 0 (Note 2c of Biggs (1993)), giving $\lfloor \frac{n}{4} \rfloor$ extra cases of the form $\lambda_i + \lambda_j = 0$.

(iv) Since $D = 2\lfloor \frac{n}{2} \rfloor$. □

These bounds are tight, e.g. for the 5×5 torus, $\epsilon = 6$, $D = 4$, and $r = 1.2$, while for the 6×6 torus, $\epsilon = 9$, $D = 6$, and $r = \frac{9}{7}$.

Corollary 8.4 *For a torus of n nodes, $r < 1 + \sqrt{n}/4$.*

Proof. If $n = m^2$, then $r \le 1 + \lfloor \frac{m}{2} \rfloor / 4 < 1 + m/4 = 1 + \sqrt{n}/4$ by Corollary 8.3. If $n = m_1 m_2$ with $m_1 > m_2$, then $r \le 1 + \lfloor \frac{m_1}{2} \rfloor \lfloor \frac{m_2}{2} \rfloor / m_1 \le 1 + m_2/4 < 1 + \sqrt{n}/4$ by Corollary 8.2. □

For nearly square tori, this bound is approached asymptotically, e.g. for the 29×30 torus, $\epsilon = 240$, $D = 29$, and $r = 8$, while $1 + \sqrt{870}/4 \approx 8.374$.

9 Cayley Graphs of Non-Abelian Groups

Many interesting graphs are Cayley graphs of non-abelian groups. We have the following corollary to Proposition 7.2:

Corollary 9.1 *Let G be a finite non-abelian group generated by S, with irreducible characters $\chi_1, \ldots, \chi_c$. Then $\Gamma(G, S)$ has:*

$$r \le \frac{\sum_i \chi_i(1)}{1 + \left\lceil \frac{\log(n-1)}{\log d} \right\rceil}$$

Proof. By Proposition 7.2 and Proposition 2.3(iv). □

For example, consider the group A_5 of even permutations on 5 elements, generated by combinations of a, b, and ab, where $a^5 = b^2 = (ab)^3 = 1$. Six Cayley graphs of A_5 are shown in Figure 1(a), 1(c), 1(g), 1(h), 1(k), and 1(l), and listed in Table 2. The group A_5 has 5 conjugacy classes, and since $60 = 1 + 3^2 + 3^2 + 4^2 + 5^2$ is a unique sum of 5 squares including 1, we do not need to know the character table in order to derive the bound $\epsilon \le 1+3+3+4+5 = 16$. We have $\lceil \log 59/\log 3 \rceil = 4$, and $\lceil \log 59/\log 4 \rceil = \lceil \log 59/\log 5 \rceil = 3$, so we can calculate the bounds on r in Table 2, which are not tight. The bounds on r are improved if we restrict our attention to *planar* Cayley graphs, which correspond either to Platonic polyhedra, prisms and antiprisms (discussed above), rings, or Archimedean polyhedra (see Section 3.10 of Babai (1996)). Consequently, for planar Cayley graphs with 60 nodes, $D \ge 12 - d$, giving the bounds in the last column of Table 2.

The group S_4 of permutations on 4 elements is generated by combinations of a, b, and ab, where $a^3 = b^2 = (ab)^4 = 1$. Four Cayley graphs of S_4 are listed

Proposition 2.3(ii)	$1 \leq r \leq \frac{n}{3}$
Regular, Proposition 2.3(iv)	$r \leq n \,/\, \left(1 + \left\lceil \frac{\log(n-1)}{\log d} \right\rceil\right)$
Strongly Regular, Proposition 2.8	$r = 1$
Distance-Transitive, Proposition 2.9	$r = 1$
n-sided Prism, Theorem 3.1	$r \leq 2\left(\lfloor \frac{n}{2} \rfloor + 1\right) / \left(\lfloor \frac{n}{2} \rfloor + 2\right) < 2$
n-sided Twisted Prism, Theorem 4.1	$r \leq 1 + 1 \,/\, \left(1 + \lfloor \frac{n}{2} \rfloor\right) \leq \frac{4}{3}$
n-sided Antiprism, Theorem 6.1	$r \leq (n+1) \,/\, \left(\lfloor \frac{n+1}{2} \rfloor + 1\right) < 2$
$m \times n$ Torus, Corollary 8.2	$r \leq 1 + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor / \left(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1\right)$
$n \times n$ Torus (odd), Corollary 8.3(ii)	$r \leq 1 + \left(\lfloor \frac{n}{2} \rfloor^2 - \lfloor \frac{n}{2} \rfloor\right) / \left(2 + 4\lfloor \frac{n}{2} \rfloor\right)$
$n \times n$ Torus (even), Corollary 8.3(iv)	$r \leq 1 + \left(\lfloor \frac{n}{2} \rfloor^2 - \lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{n}{4} \rfloor\right) / \left(2 + 4\lfloor \frac{n}{2} \rfloor\right)$
Cayley graph, Corollary 9.1	$r \leq \left(\sum_i \chi_i(1)\right) / \left(1 + \left\lceil \frac{\log(n-1)}{\log d} \right\rceil\right)$
Planar Cayley graph, Corollary 9.2	$r < 2$
Random graph, Corollary 10.3	$r \geq n \,/\, \left(4 + \left\lceil \frac{\log n + 6}{\log \log n} \right\rceil\right)$

Table 4: Summary of Bounds on Symmetry Ratio

in Table 3. S_4 has 5 conjugacy classes, and since $24 = 1 + 1 + 2^2 + 3^2 + 3^2$ is a unique sum of 5 squares including 1, we do not need to know the character table in order to derive the bound $\epsilon \leq 1 + 1 + 2 + 3 + 3 = 10$. This bound on ϵ is tight for the truncated octahedron, and hence so is the planar bound on r.

In fact, by checking the remaining Archimedean polyhedra which are Cayley graphs, we obtain a more general bound for planar Cayley graphs:

Corollary 9.2 *For all finite planar Cayley graphs, $r < 2$.*

Proof. By Proposition 2.9, Theorem 3.1, Theorem 6.1, and checking the remaining cases. □

10 Random Graphs

Sufficiently connected random graphs have the maximum number of distinct eigenvalues $\epsilon = n$, since the adjacency matrix is uncorrelated:

Conjecture 10.1 *Consider a graph of n nodes formed by randomly adding sufficient edges so that the graph is connected, but $d_{\text{ave}} \leq \frac{n}{2}$. Then as $n \to \infty$, the probability approaches 1 that $\epsilon = n$.*

This seems to be a "folk theorem," with no formal proof in the literature. Since the probability of non-trivial automorphisms vanishes (Theorem 9.4 of Bollobás (2001)), the problem is with linearly dependent rows, and the major difficulty is with nodes of low degree. But by Theorem 7.3 of Bollobás (2001), $d_{\text{ave}} = \log n$ at the point of connectedness, and since degrees follow a Poisson distribution, the number of nodes of degree at most k is bounded by a multiple of $(\log n)^k$, and so (for some p and q) the expected number of linearly dependent low-degree rows in the adjacency matrix is $p(\log n)^q / n \to 0$.

Alternatively, it may be possible to adapt similar results from random matrix theory (Lévêque 2004).

Empirically the result holds. For example, we generated a sample of 534 random 60-node connected graphs with average degree $4 \leq d_{\text{ave}} \leq 9$ and all had $\epsilon = 60$. An additional sample of 73 random 600-node connected graphs all had $\epsilon = 600$. With regards to the symmetry ratio, we have the following:

Proposition 10.2 *Consider a graph of n nodes formed by randomly adding sufficient edges so that the graph is connected. Then as $n \to \infty$, the probability approaches 1 that:*

$$D \leq 3 + \left\lceil \frac{\log n + 6}{\log \log n} \right\rceil$$

Proof. By Theorem 10.17 of Bollobás (2001). □

Corollary 10.3 *Consider a graph of n nodes formed by randomly adding sufficient edges so that the graph is connected, but $d_{\text{ave}} \leq \frac{n}{2}$. Then as $n \to \infty$, the probability approaches 1 that:*

$$r \geq \frac{n}{4 + \left\lceil \frac{\log n + 6}{\log \log n} \right\rceil}$$

For $n = 60$, these results give $D \leq 11$ and $r \geq 5$, and indeed for our 534 random 60-node graphs, $3 \leq D \leq 9$, and hence $6 \leq r \leq 15$. For $n = 600$, we get $D \leq 10$ and $r \geq 54$, and for our 73 random 600-node graphs, $5 \leq D \leq 7$ and $75 \leq r \leq 100$. For comparison, a 600-node torus has $r < 1 + \sqrt{n}/4 \approx 7.124$. Random graphs are thus associated with very high values of r.

11 Conclusions

In this paper, we have introduced a measure of symmetry which we call the *symmetry ratio* of a network, defined to be $r = \epsilon / (D + 1)$, where ϵ is the number of distinct eigenvalues of the network, and D is the diameter. Simulation experiments have shown that the symmetry ratio has utility in partially predicting the robustness of a network in the face of attack. We have proved several bounds on the symmetry ratio, summarised in Table 4, and considered a number of examples, shown in Figures 1 and 4, and Tables 2 and 3. The networks we have considered fall into five families:

(i) Highly symmetrical (e.g. distance-transitive) networks, with $r = 1$.

(ii) Prism-like networks and planar Cayley graphs, with $r < 2$.

(iii) Tori with $r < 1 + \sqrt{n}/4$ (Corollary 8.4).

(iv) Non-planar Cayley graphs of non-abelian groups, where r may be higher.

(v) Non-regular (e.g. random) networks, where r may be much higher.

The proof techniques that we have used in this paper can also be applied to finding bounds on the symmetry ratio for other classes of network.

12 Acknowledgements

The authors are indebted to Richard Taylor and to three anonymous referees for comments on an earlier draft of this paper.

References

Babai, L. (1979), 'Spectra of Cayley graphs', *Journal of Combinatorial Theory, Series B* **27**, 180–189.

Babai, L. (1996), Automorphism groups, isomorphism, reconstruction, *in* R. L. Graham, M. Groetschel & L. Lovasz, eds, 'Handbook of Combinatorics', Elsevier. (also Technical Report TR-94-10, University of Chicago, available electronically at `www.cs.uchicago.edu`).

Biggs, N. (1993), *Algebraic Graph Theory*, 2nd edn, Cambridge University Press.

Bollobás, B. (2001), *Random Graphs*, 2nd edn, Cambridge University Press.

Dekker, A. H. (2005), Simulating network robustness for critical infrastructure networks, *in* V. Estivill-Castro, ed., 'Proceedings of 28th Australasian Computer Science Conference', Vol. 38 of *Conferences in Research and Practice in Information Technology*, Newcastle, Australia.

Dekker, A. H. & Colbert, B. (2004*a*), Network robustness and graph topology, *in* V. Estivill-Castro, ed., 'Proceedings of 27th Australasian Computer Science Conference', Vol. 26 of *Conferences in Research and Practice in Information Technology*, Dunedin, New Zealand, pp. 359–368. Available electronically at `crpit.com/confpapers/CRPITV26Dekker.pdf`.

Dekker, A. H. & Colbert, B. (2004*b*), Scale-free networks and robustness of critical infrastructure networks, *in* 'Proceedings of 7th Asia-Pacific Conference on Complex Systems', Cairns, Australia.

Dodge, M. (2004), 'An atlas of cyberspaces'. `www.cybergeography.org/atlas/`.

Gibbons, A. (1985), *Algorithmic Graph Theory*, Cambridge University Press.

James, G. & Liebeck, M. (1993), *Representations and Characters of Groups*, Cambridge University Press.

Lévêque, O. (2004), 'Lecture notes on random matrices and communication systems'. `lthiwww.epfl.ch/~leveque/Matrices/`.

Murty, M. R. (2003), 'Ramanujan graphs', *J. Ramanujan Math. Soc.* **18**(1), 1–20. Available at `www.mast.queensu.ca/~murty/ramanujan.pdf`.