Average Distance as a Predictor of Synchronisability in Networks of Coupled Oscillators

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Abstract

The importance of networks of coupled oscillators is widely recognized. Such networks occur in biological systems like the heart, in chemical systems, in computational problems, and in engineering systems. Systems of coupled oscillators can also be used as an abstract model for synchronisation in organisations. Here we show that synchronisability in a specific coupled-oscillator model, the Kuramoto model, is best predicted using the average distance (or characteristic path length) between nodes in the network. We do this by simulating the Kuramoto dynamics on a collection of networks of varying type, including Random, Small-World, and Scale-Free networks. Furthermore, we show that, for several real-world networks, a simple estimate based on the average distance can predict the coupling required for networks to synchronise within a threshold time.

Keywords: Kuramoto model, coupled oscillators, synchronisation, network, average distance.

1 Introduction

The importance of networks of coupled oscillators is widely recognized (Strogatz, 2003). Such networks occur in biological systems like the heart (Winfree, 1980), in chemical systems (Kuramoto, 1948), in computational problems (Lee and Lister, 2008), and in engineering systems (Olfati-Saber *et al.*, 2007). Systems of coupled oscillators can also be used as an abstract model for synchronisation in organisations (Dekker, 2007a; Kalloniatis, 2008).

The Kuramoto model (Strogatz, 2000; Dorogovtsev *et al.*, 2008) is a simple system of n coupled oscillators $O_1 \ldots O_n$, each with a natural frequency f_i (assumed to come from a unimodal and symmetric distribution) and a phase angle θ_i (Figure 1 provides an example). Phase angles change so as to become closer to those of neighbouring oscillators, according to the differential equation:

$$\theta_i' = f_i + k \sum_{j}^{i \leftrightarrow j} \sin(\theta_j - \theta_i) \qquad (1)$$

where the sum is taken over all oscillators O_j connected to O_i in the network (this equation is sometimes also written with the coupling constant k scaled by a factor of n).

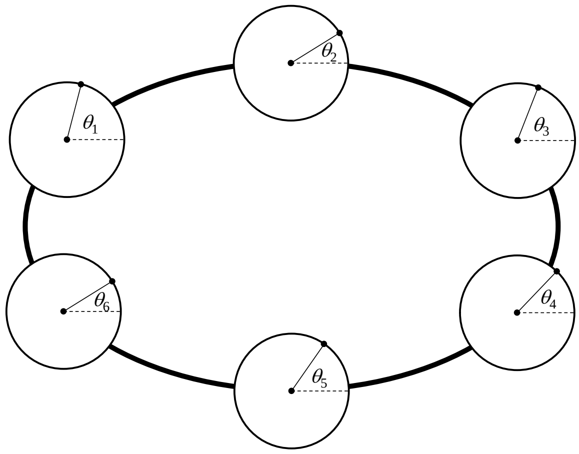

Figure 1: Six Kuramoto oscillators connected in a ring network, showing the phase angle θ_i of each oscillator.

The degree of synchronisation in the network is measured by the correlation r between the phases of all the oscillators (Strogatz, 2000), which is equal to 1 when all the phases are identical:

$$r = \sqrt{\left(\frac{\sum_i \sin\theta_i}{n}\right)^2 + \left(\frac{\sum_i \cos\theta_i}{n}\right)^2} \qquad (2)$$

In the case of a completely connected network, Kuramoto showed that the oscillators began to synchronise as the coupling constant k in Equation (1) exceeded a critical threshold (Strogatz, 2000):

$$k_C = \frac{2}{ng(f_0)\pi} \qquad (3)$$

where $g()$ is the probability density function for frequencies, and f_0 is the centre or peak of the distribution. In the case that the frequencies are taken from a uniform distribution of width w (i.e. uniformly from f_0–w/2...f_0+w/2), Equation (3) reduces to:

$$k_C = \frac{2w}{n\pi} \qquad (4)$$

For more general networks, analytical results are limited (Strogatz, 2000). Nevertheless, a simple optimistic (low) estimate for the critical coupling can be made that assumes that purely local coupling is sufficient for synchronisation. This estimate simply scales Equation (4) by n/d, where d is the average degree (average number of links per node):

$$k_L = \frac{2w}{d\pi} \quad (5)$$

A more pessimistic (higher) estimate can be made, noting that every pair of oscillators is in fact connected, not necessarily by a link, but by a path of average distance D. Since the coupling attenuates by a factor of k for each link in the path, this yields:

$$k_H^D = \frac{2w}{n\pi} \quad (6)$$

or:

$$k_H = \sqrt[D]{\frac{2w}{n\pi}} \quad (7)$$

For large networks, this equation yields reasonable values for k_H only when average distances in the network are small, in the sense of scaling logarithmically with n. This is consistent with the importance of the "small world" effect identified by Watts and Strogatz (1998).

Jadbabaie *et al*. (2004) conducted a stability analysis on Equation (1) and derived a lower bound on the critical coupling of:

$$k_J = \frac{2\|f\|_2}{\lambda_2\sqrt{n}} \quad (8)$$

where $\|f\|_2$ is the 2-norm of the frequency distribution, and λ_2 is the so-called *algebraic connectivity*: the smallest non-zero eigenvalue of the Laplacian matrix for the network (the use of the Laplacian matrix arises naturally in stability analysis).

It must be noted that, for some networks, synchronisation is not guaranteed even for very high couplings k. In particular, ring-like networks have stable locally synchronised solutions (Winfree, 1980) where, for small integers m:

$$\theta_i = \theta_0 + \frac{2mi\pi}{n} \quad (9)$$

So that, for example, $\sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i) = 0$. The system may fail to synchronise globally by becoming "trapped" in such a locally synchronised state. Similar locally synchronised solutions occur for spherical networks, such as the one shown in Figure 2.

Although the stability of the globally synchronised state is important, it is equally important to consider the *synchronisability* of the system – the ease with which the globally synchronised state is achieved (if at all). As a measure of the synchronisability, we take the median time t to reach a correlation of $r = 0.9$ (in our simulations of the Kuramoto model, all systems having reached this point continue to have r asymptotically approach 1). We calculate the median over 101 simulation runs with coupling constant $k = 0.1$ and with different (randomly chosen) initial phase angles θ_i and frequencies f_i (the latter from a uniform distribution of width $w = 0.001$). The use of the median allows for some runs becoming "trapped" in locally synchronised states.

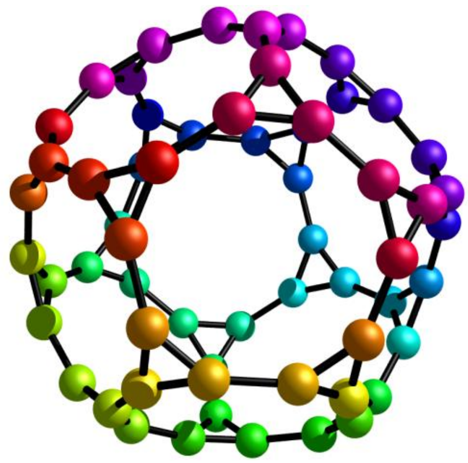

Figure 2: Locally synchronised state for a spherical (truncated dodecahedron) network. Phase angles for each oscillator are shown using colours on a colour wheel.

2 Simulation Experiments

In our simulation experiment, we discretised Equation (1) using 100 steps per time unit, with a varied sample of 79 networks, all with $n = 60$ oscillators:

- 40 Random (Erdős-Rényi) networks (Bollobás, 2001), with average degrees d ranging from 3 to 10;
- 20 Scale-Free (preferential-attachment) networks (Albert and Barabási, 2002; Barabási, 2002), with average degrees d ranging from 2 to 5;
- 10 Small-World networks generated by the Watts rewiring process (Watts and Strogatz, 1998; Watts, 2003), with a probability $p = 0.1$ of rewiring a link;
- 1 social network resulting from a survey of informal communication within an organisation (Dekker, 2007b), with average degree $d = 4$; and
- the 8 networks in Figure 3, including 1 tree, 1 torus, and 6 spherical networks.

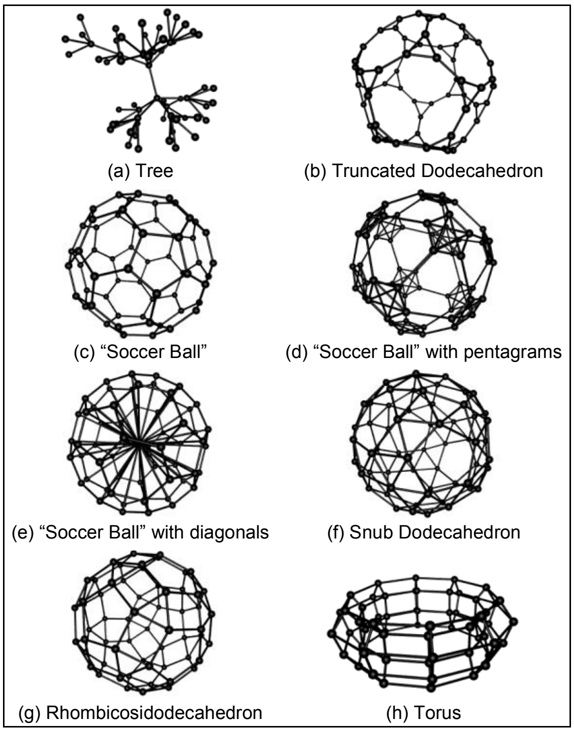

Figure 3: Eight of the sample networks used.

For this list of networks, $k_L \leq 0.0003$, $k_J \leq 0.003$, and $0.003 \leq k_H \leq 0.12$. The selected coupling constant $k = 0.1$ would therefore be expected to result in synchronisation difficulties for some of the networks, and hence is a good value for exploring synchronisability.

Figure 4 shows the result of the simulation experiment. The average distance D in the network turns out to be an excellent predictor of synchronisability. Linear regression on log-transformed data gives a best-fit power law:

$$t = 0.44 D^{4.0} \qquad (10)$$

with R^2 = 93%. In contrast, the algebraic connectivity λ_2 is less effective as a predictor (R^2 = 84%), and similarly for the average degree d (R^2 = 68%) and the clustering coefficient (Watts and Strogatz, 1998) (R^2 = 3%).

Towards the top of Figure 4, the fit to Equation (10) becomes less good, and systematic differences appear between different types of network. The tree in Figure 3(a), the Scale-Free networks of average degree 2, and the Small-World networks take about 110 time units longer to synchronise than the other networks (statistically significant at the 0.001 level, by analysis of variance). There are two reasons for this difference. On the one hand, the Scale-Free networks of average degree 2 are essentially just trees and, like the tree in Figure 3(a), simply do not have enough links to synchronise well. The Small-World networks, on the other hand, retain enough of the original ring-like structure to permit distorted versions of the locally synchronised solution, which interferes with global synchronisation. In both cases, the result is poorer performance than Equation (10) would predict.

The networks which synchronise most effectively are the Random (Erdős-Rényi) networks of high degree. However, if we assign a cost to links, by minimising $d\sqrt{t}$, then the most efficiently synchronising networks are the Scale-Free networks with average degree $d \geq 3$.

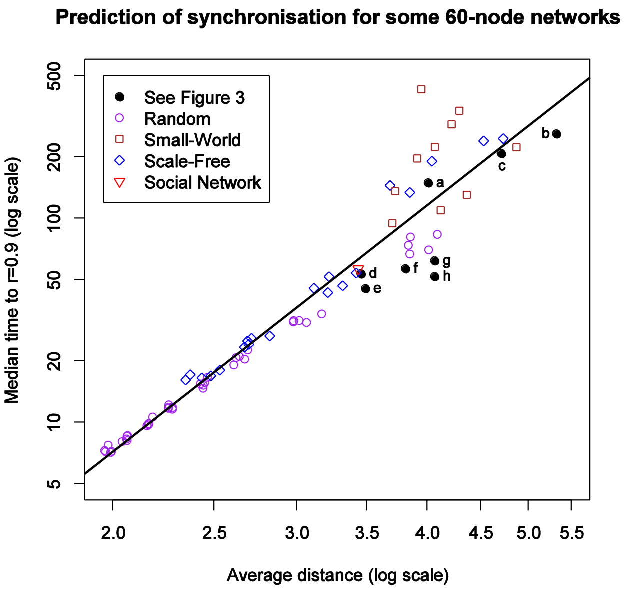

Figure 4: Experimental results for the set of 79 networks. The median time t to r = 0.9 can be predicted quite well by 0.44 $D^{4.0}$.

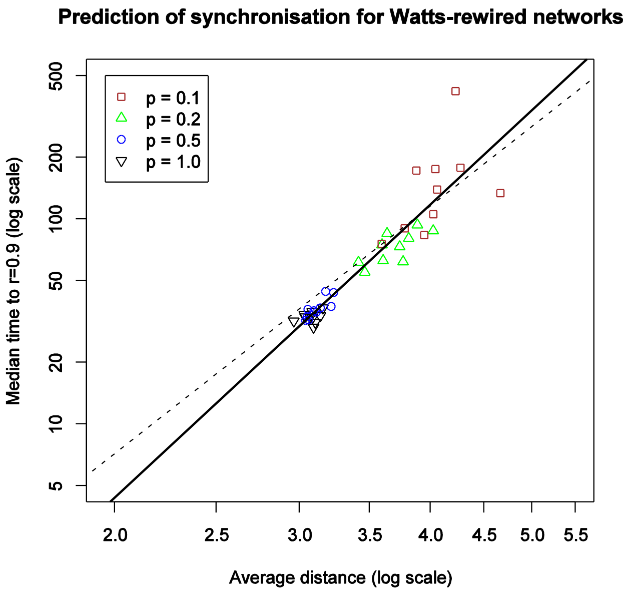

Figure 5: Experimental results for 40 Watts-rewired networks. The best-fit power law is 0.16 $D^{4.8}$, somewhat steeper than the power law of Figure 4 (which is shown dashed here for comparison).

The fact that Scale-Free networks with average degree $d \geq 3$ are the most efficient synchronisers may help to explain the prevalence of Scale-Free networks in biological structures (Albert and Barabási, 2002).

In a second experiment, we explored the synchronisation of Watts-rewired networks in more detail, using 40 networks generated by the Watts rewiring process (Watts and Strogatz, 1998; Watts, 2003), with the probability p of rewiring a link ranging from 0.1 to 1.0. Linear regression on log-transformed data gives a best-fit power law:

$$t = 0.16 D^{4.8} \qquad (11)$$

with R^2 = 85%. As before, the algebraic connectivity λ_2 is less effective as a predictor (R^2 = 54%). As the rewiring reduces the average distance and destroys the original ring structure, synchronisability continuously improves, at least until p = 0.5, at which point the Watts process produces networks very similar to Erdős-Rényi random networks. Figure 5 illustrates the results.

In a third experiment, we explored the robustness of synchronisability using two 60-node Random networks (with average degree d = 5 and d = 7 respectively, and two Scale-Free networks (with d = 4 and d = 5). We repeatedly removed the most "central" node, using the definition of centrality in Dekker (2005), and recalculated the median time to r = 0.9. As usual, the Scale-Free networks were less resistant to such targeted attacks (Albert and Barabási, 2002; Dekker and Colbert 2004), becoming disconnected after just three node removals, while the Random networks could absorb at least ten node removals. As Figure 6 shows, the time to synchronisation is predicted extremely well by the average distance (R^2 = 99%), with a power law:

$$t = 0.82 D^{3.3} \qquad (12)$$

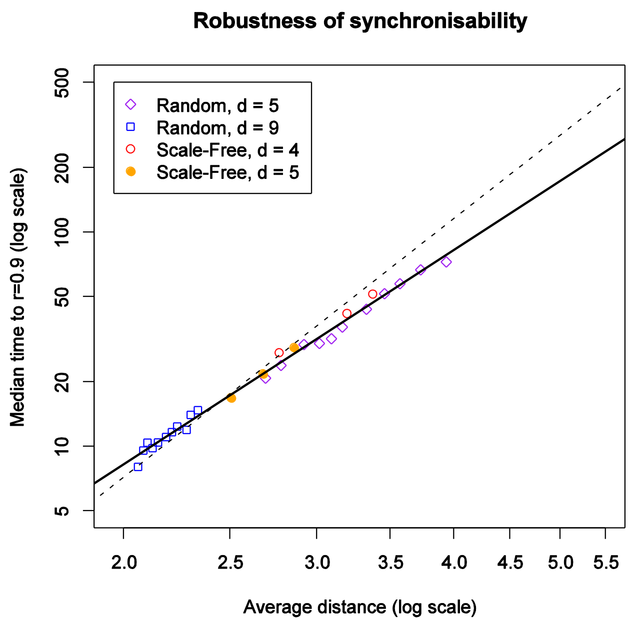

Figure 6: Experimental results for node removals. The best-fit power law is 0.82 $D^{3.3}$, somewhat shallower than the power law of Figure 4 (which is shown dashed here for comparison). The spacing of the data points shows that, for the Scale-Free networks, each node removal has a larger effect on average distance (and hence on synchronisability) than for the Random networks.

The spacing of the data points in Figure 6 shows that, for the Scale-Free networks, each node removal has a larger effect on average distance and hence on synchronisability (on average, an increase of 8.6% for the Scale-Free networks per node removal, compared to 2.6% for the Random networks, a difference significant at the 0.0001 level). Scale-Free networks are therefore efficient, but not robust, synchronisers.

In a final experiment, we considered synchronisability of six real-world networks:

- a communications network for the southeast USA (Dodge, 2004), with $n = 21$ and $D = 4.33$;
- a connected subset of a scientific coauthorship network (Newman, 2006), with $n = 57$ and $D = 3.66$;

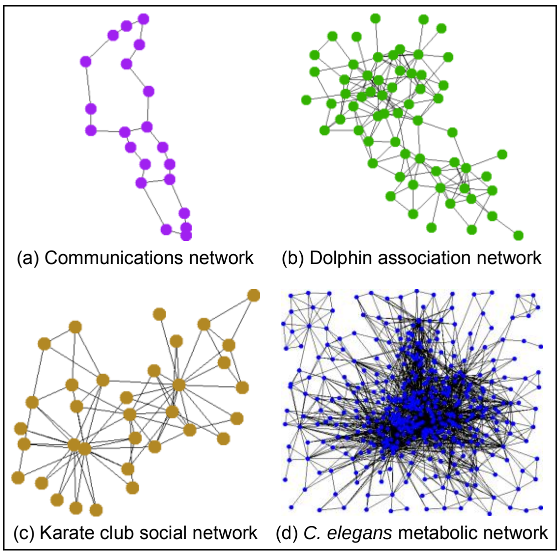

Figure 7: Four of the real-world networks used in the fourth experiment.

- an association network between dolphins in a community living off Doubtful Sound, New Zealand (Lusseau *et al*., 2003), with $n = 62$ and $D = 3.36$;
- a social network from a karate club at a US university (Zachary, 1977), with $n = 34$ and $D = 2.41$;
- the (Scale-Free) metabolic network of the nematode *Caenorhabditis elegans* (Duch and Arenas, 2005), with $n = 453$ and $D = 2.66$; and
- a food web for Chesapeake Bay (Baird and Ulanowicz, 1989) considered as an undirected network, with $n = 39$ and $D = 1.84$.

Figure 7 illustrates four of these networks.

For a frequency distribution width $w = 0.001$ and a varying coupling constant k, Figure 8 shows that the median time t to $r = 0.9$ was inversely proportional to k in each case, as would be expected. Furthermore, the pessimistic critical coupling estimate k_H (7) was in each case a reasonable predictor of the coupling needed to achieve a threshold synchronisation time of $t = 200$ (with $R^2 = 94\%$ for the match between prediction and actuality). This estimate therefore provides a useful guideline for assessing the synchronisability of networks.

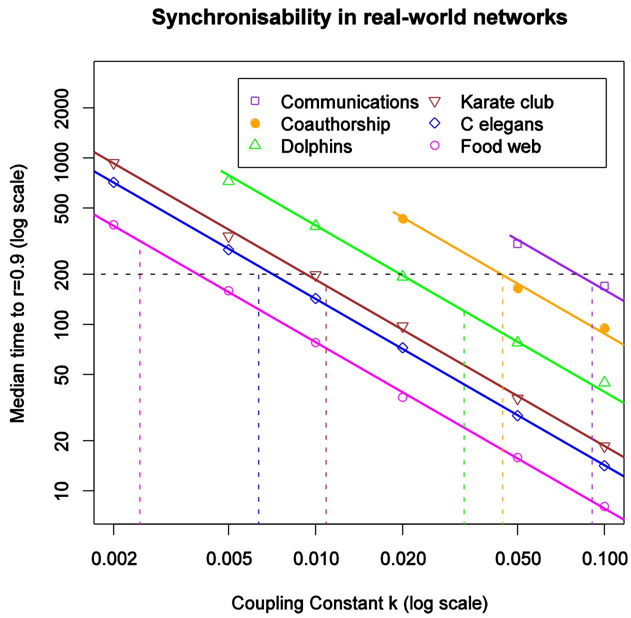

Figure 8: Experimental results for third experiment. The median t to $r = 0.9$ is inversely proportional to the coupling constant k, and the pessimistic critical coupling estimate k_H (dashed lines) is in each case a reasonable predictor of the coupling needed to achieve $t = 200$. Missing data points represent values of k for which synchronisation did not occur within a limit of 1000 time units.

3 Discussion

The simulation experiments reported here have implications for the design of any networked system which is expected to synchronise – in particular, for the design of organisational structures (Dekker, 2007a), multi-agent systems (Olfati-Saber *et al.*, 2007), and computer networks.

Our results underscore the importance of a low average distance for the underlying network topology – the "small world" condition (Watts and Strogatz, 1998).

However, the importance of a low average distance raises the question: how low is low enough? The pessimistic critical coupling estimate k_H which we have introduced here provides an indication of whether the average distance in a network is sufficiently low. Substituting $w = 0.001$ in equation (7) for the networks in our fourth experiment gives values of k_H in the range 0.0025 to 0.091. For comparison, the network for the Western States Power Grid studied by Watts and Strogatz (1998) has $n = 4{,}941$ and $D = 18.7$, giving $k_H = 0.43$. This rather high value suggests that the power grid could be subject to synchronisation difficulties – so that the 1996 West Coast power outages (Neumann, 1996) are not altogether surprising.

Scale-Free networks with average degree $d \geq 3$ are efficient synchronisers, which may help to explain the prevalence of Scale-Free networks in biological structures (Albert and Barabási, 2002). However, the synchronisability of Scale-Free networks is not robust against the targeted removal of nodes.

4 References

Albert, R. and Barabási, A.-L. (2002), "Statistical mechanics of complex networks," *Reviews of Modern Physics*, **74**, 47–97.

Baird, D. and Ulanowicz, R.E. (1989), "The seasonal dynamics of the Chesapeake Bay ecosystem," *Ecol. Monogr.*, **59**, 329–364.

Barabási, A.-L. (2002), *Linked: The New Science of Networks*, Perseus Publishing, Cambridge, MA.

Ben-Naim, E., Frauenfelder, H., and Toroczkai, Z (2004), *Complex Networks*, Springer, Berlin.

Bollobás, B. (2001), *Random Graphs*, 2nd edition Cambridge University Press, Cambridge, UK.

Dekker, A.H. and Colbert, B. (2004), "Scale-Free Networks and Robustness of Critical Infrastructure Networks," *Proceedings of the 7th Asia-Pacific Conference on Complex Systems*, Cairns, December, 685–699: www.complexity.org.au/ci/vol12/msid04/

Dekker, A.H. (2005), "Conceptual Distance in Social Network Analysis," *Journal of Social Structure*, **6**(3): www.cmu.edu/joss/content/articles/volume6/dekker/

Dekker, A.H. (2007a), "Studying Organisational Topology with Simple Computational Models," *Journal of Artificial Societies and Social Simulation*, **10**(4): jasss.soc.surrey.ac.uk/10/4/6.html

Dekker, A.H. (2007b), "Realistic Social Networks for Simulation using Network Rewiring," *MODSIM International Congress on Modelling and Simulation*, Oxley, L. and Kulasiri, D. (eds), 677–683: www.mssanz.org.au/MODSIM07/papers/13_s20/RealisticSocial_s20_Dekker_.pdf

Dodge, M., (2004), "An Atlas of Cyberspaces," www.cybergeography.org/atlas/more_isp_maps.html

Dorogovtsev S.N., Goltsev, A.V., and Mendes, J.F.F. (2008), "Critical phenomena in complex networks," *Reviews of Modern Physics*, **80**(4): arXiv:0705.0010v2

Duch, J. and Arenas, A. (2005), "Community identification using Extremal Optimization," *Physical Review E*, **72**, 027104.

Jadbabaie , A., Motee , N., and Barahona , M. (2004), "On the Stability of the Kuramoto Model of Coupled Nonlinear Oscillators," *American Control Conference*.

Kalloniatis, A. (2008), "From Kuramoto to Boyd: applying networked dynamical systems to military Command and Control," *International Workshop on Complex Systems and Networks*, Canberra: labnetcon.anu.edu.au/IWCSN/Lecture_Slides/2008-10-03-Kalloniatis.pdf

Kuramoto, Y. (1948), *Chemical Oscillations, Waves, and Turbulence*, Springer, Berlin.

Lee, S. and Lister, R. (2008), "Experiments in the Dynamics of Phase Coupled Oscillators When Applied to Graph Colouring," 31st Australasian Computer Science Conference (ACSC), Wollongong, Dobbie, G. and Mans, B. (eds), *Conferences in Research and Practice in Information Technology*, **74**, 83–89: crpit.com/confpapers/CRPITV74Lee.pdf

Lusseau, D., Schneider, K., Boisseau, O.J., Haase, P., Slooten, E., and Dawson, S.M. (2003), "The bottlenose dolphin community of Doubtful Sound features a large proportion of long-lasting associations," *Behavioral Ecology and Sociobiology*, **54**, 396–405.

Neumann, P.G. (1996), "Western U.S. power blackout," *Risks Digest*, **18**(25): catless.ncl.ac.uk/risks/18.25.html

Newman, M.E.J. (2006), "Finding community structure in networks using the eigenvectors of matrices," *Phys. Rev. E*, **74**, 036104.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007), "Consensus and Cooperation in Networked Multi-Agent Systems," *Proceedings of the IEEE*, **95**(1), 215–233.

Strogatz, S.H. (2000), "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D: Nonlinear Phenomena*, **143**, 1–20.

Strogatz, S. (2003), *Sync: The Emerging Science of Spontaneous Order*, Hyperion, New York.

Watts, D.J. and Strogatz, S.H. (1998), "Collective dynamics of 'small world' networks," *Nature*, **393**, 440–442.

Watts, D.J. (2003), *Six Degrees: The Science of a Connected Age*, William Heinemann, London.

Winfree, A.T. (1980), *The Geometry of Biological Time*, Springer, New York.

Zachary, W.W. (1977), "An information flow model for conflict and fission in small groups," *Journal of Anthropological Research*, **33**, 452–473.